Undergraduate Applied Analysis One

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Contents

1 Preliminaries

There are three axioms that we start with. The Field Axiom, the Positivity Axiom, and the Completeness Axiom.

1.1 Field Axiom

To explain the field axiom we will use an example. $\mathbb{R}$ is a field as it obeys the definition of a field.

Definition 1.1 (Field)**.** A field, $\mathcal{F}$, is a set F together with two operations, "+" and "·", $\mathcal{F} = (F, +, \cdot)$ such that for all $a, b, c, \in F$:
If $x, y \in \mathcal{F}$ then
A0: Closure under Addition: $x + y \in \mathcal{F}$.
A1: Communitivity under Addition: $x + y = y + x$.
A2: Associativity under Addition: $x + (y + z) = (x + y) + z$.
A3: Additive Identity: $\exists 0 \in \mathcal{F}$ such that $x + 0 = x$.
A4: Additive Inverse: $\forall x \in \mathcal{F}$, $\exists (-x) \in \mathcal{F}$ such that $x + (-x) = 0$.
M0: Closure under Multiplication: $xy \in \mathcal{F}$.
M1: Communitivity under Multiplication: $xy = yx$.
M2: Associativity under Multiplication: $x(yz) = (xy)z$.
M3: Multiplicative Identity: $\exists 1 \in \mathcal{F}$ such that $(1)x = x$.
M4: Multiplicative Inverse: $\forall x \in \mathcal{F}$, $\exists x^{-1}$ such that $(x)\left(x^{-1}\right) = 1$ as long as $x \neq 0$.
C1: Distributive Property: $x(y + z) = xy + xz$.
C2: Non-Triviality: $1 \neq 0$

By this definition, $\mathbb{R}$ is a field.

1.2 Positivity Axiom

The following two postulates define the Positivity Axiom.

Definition 1.2 (Positivity)**.** 1. If a and b are positive $(a, b > 0)$ then so is $a + b$ and ab.

2. For $a \in \mathbb{R}$, exactly one of the following is true:
   (a) $a > 0$
   (b) $a < 0$
   (c) $a - 0$

These postulates give an ordering on $\mathbb{R}$. Not all fields have order however. $\mathbb{C}$ for example does not have this order.

1.3 Completeness Axiom

A nonempty set $S \subset \mathbb{R}$ is bounded above if there is a number c such that $x \leq c, \forall x \in S$. c is called the upper bound of S.

Definition 1.3 (Completeness)**.** If S is a non-empty subset of $\mathbb{R}$ that is bounded above then, among the set of upper bounds of S, there exists a smallest, or least upper bound (lub, supremum, sup).

A non-empty set $S \subset \mathbb{R}$ is bounded below if $\exists c$ such that $c \leq x, \forall x \in S$ then c is called the greatest lower bound (glb, infinum, inf).

Definition 1.4 (Maximum)**.** If $S \subset \mathbb{R}, S \neq \emptyset, c \in S$, c is called the maximum of S provided that c is an upper bound.

1.4 Induction

Definition 1.5 (Inductive)**.** A set S of real numbers is inductive if

1. $1 \in S$
2. $x \in S \Rightarrow x + 1 \in S$

We can prove using induction, which is covered in my other notes for Discrete Math.

1.5 Denseness

Definition 1.6 (Dense)**.** A set S is said to be dense in $\mathbb{R}$ provided every interval $I = (a, b)$ where $a < b$ contains a member of S.

1.6 Useful Formulas

1.6.1 Distribution of Integers

For any c there exists exactly one integer in the interval $[c, c + 1)$.

For any a, b with $a < b$ there exists a rational number in the interval (a, b).

1.6.2 Archimedean Property

The following two statements are equivalent.

- For any $c > 0$, there exists some $n \in \mathbb{N}$ such that $n > c$.
- For any $\epsilon > 0$, there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

1.6.3 Formulas

- Difference of Powers:
$$a^n - b^n = (a - b) \sum_{k=0}^{n-1} a^{n-1-k} b^k$$
- Geometric Sum
$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$
- Binomial Formula[1]
$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

[1]
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

2 Convergent Sequences

Definition 2.1 (Sequence)**.** A sequence of real numbers is a real valued function, $f(x)$, whose domain is the set of natural numbers.

Definition 2.2 (Convergence)**.** A sequence $\{a_n\}$ is said to converge to a provided that for every positive number ϵ there exists an N such that

$$|a_n - a| < \epsilon \qquad \forall n \geq N$$

In other words, $\{a_n\}$ converges to a if $\forall \epsilon$ there exists an N such that

$$a - \epsilon < a_n < a + \epsilon$$

If $\{a_n\}$ converges to a then

$$\lim_{n \to \infty} a_n = a$$

Definition 2.3 (Comparison Lemma)**.** Let $\{a_n\}$ converge to a, then $\{b_n\}$ converges to b if there exists a non-negative number C and N such that

$$|b_n - b| \leq C|a_n - a| \qquad \forall n \geq N$$

2.1 Properties of Convergent Sequences

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences that converge to a and b respectively.

- *Sum Property:*
$$\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n$$
- *Constant Multiple:*
$$\lim_{n \to \infty} \alpha \cdot a_n = \alpha \cdot a$$
- *Product Property:*
$$\lim_{n \to \infty} a_n \cdot b_n = a \cdot b$$
- *Inverse:*
$$\lim_{n \to \infty} \frac{1}{b_n} = \frac{1}{b}$$
- *Quotient Property:*
$$\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n}$$
- *Linearity Property:*
$$\lim_{n \to \infty} (\alpha \cdot a_n + \beta \cdot b_n) = \alpha \lim_{n \to \infty} a_n + \beta \lim_{n \to \infty} b_n$$
- *Polynomial Property:* For any polynomial $p : \mathbb{R} \to \mathbb{R}$
$$\lim_{n \to \infty} p(a_n) = p(a)$$

- *Convergence:* A sequence is bounded if

$$|\{a_n\}| \leq M$$

Every convergent sequence is bounded.

2.2 Denseness

Definition 2.4 (Dense)**.** A subset S is dense if for every open interval (a, b) there exists some point in S contained in the interval.

We need to establish the concept of having a sequence being contained in a set. Some set S, $\{x_n\}$ is in the set S provided that for all indices n, $x_n \in S$.

Therefore S is dense in $\mathbb{R}$ if and only if every x is the limit of some sequence in S.

Definition 2.5 (Sequential Density)**.** Every number is the limit of a sequence of rational numbers.

Definition 2.6 (Closed)**.** $S \subseteq \mathbb{R}$ is closed if $\{a_n\}$ is a sequence in S that converges to a and the limit a is contained in S.

2.3 Monotone Sequences

A sequence $\{a_n\}$ is monotonically increasing if $a_{n+1} \geq a_n$, and decreasing if $a_{n+1} \leq a_n$.

Theorem 2.1 (Monotone Convergence Theorem)**.** *A monotone sequence converges if and only if it is bounded.*

The following propositions hold from the Monotone Convergence Theorem.

- Let c be a number such that $|c| < 1$, then

$$\lim_{n \to \infty} c^n = 0$$

Theorem 2.2 (Nested Interval Theorem)**.** *For each natural number n, let a_n and b_n be numbers such that $a_n < b_n$, and consider the interval $I_n \equiv [a_n, b_n]$. Assume that $I_{n+1} \subseteq I_n$ for every index n. Also assume that $\lim_{n\to\infty}[b_n - a_n] = 0$. Then there is exactly one point x that belongs to the interval I_n for all n, and both of the sequences $\{a_n\}$ and $\{b_n\}$ converge to this point.*

2.4 The Sequential Compactness Theorem

Definition 2.7 (Subsequence)**.** Let $\{a_n\}$ be some sequence. Let $\{n_k\}$ be a sequence of natural numbers that is strictly increasing. Then the sequence $\{b_k\}$ defined by $b_k = a_{n_k}$ for any index k is called a subsequence of the sequence $\{a_n\}$.

The following properties follow:

- If $\{a_n\}$ converges to a, then every subsequence also converges to the same limit a.

Theorem 2.3 (Monotone Subsequences)**.** *Every sequence has a monotone subsequence.*

Theorem 2.4 (Subsequence Convergence)**.** *Every bounded sequence has a convergent subsequence.*

Definition 2.8 (Sequential Compactness)**.** A set of real numbers S is said to be sequentially compact provided that every sequence $\{a_n\}$ in S has a subsequence that converges to a point that belongs to S.

2.5 Covering Properties of Sets

Let S be a subset of $\mathbb{R}$ that is closed and bounded. Then S is sequentially compact.

Definition 2.9 (Compact)**.** A subset S of $\mathbb{R}$ is said to be compact provided that any cover of S by a collection $\{I_n\}_{n=1}^{\infty}$ of open intervals has a finite subcover, that is, if for each index n, I_n is an open interval and

$$S \subseteq U_{n=1}^{\infty} I_n$$

then there is an index N such that

$$S \subseteq U_{n=1}^{\infty} I_N$$

Let S be a compact subset of $\mathbb{R}$. Then S is both closed and bounded.
Let S be a sequentially compact subset of $\mathbb{R}$. Then S is compact.

Theorem 2.5 (Sequentially Compact Properties)**.** *For a subset S of $\mathbb{R}$, the following three assertions are equivalent.*

- *S is closed and bounded.*
- *S is sequentially compact.*
- *S is compact.*

3 Continuity

In essence, a function is continuous if there exists a sequence in x that corresponds to a sequence in y.

Definition 3.1 (Continuous)**.** A function $f : D \to \mathbb{R}$ is said to be continuous at the point x_0 in D provided that whenever $\{x_n\}$ is a sequence in D that converges to x_0, the image sequence $\{f(x_n)\}$ converges to $f(x_0)$. The function $f : D \to \mathbb{R}$ is said to be continuous provided that it is continuous at every point in D.

Given two functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$, we define the sum $f + g : D \to \mathbb{R}$, and the product $fg : D \to \mathbb{R}$ by

$$\begin{aligned} (f+g)(x) &\equiv f(x) + g(x) \\ (fg)(x) &\equiv f(x)g(x) \end{aligned}$$

Some base properties hold.

1. Sums are continuous
2. Products are continuous
3. For non-zero functions, the quotient is continuous.
4. Polynomial quotients are continuous.
5. Compositions are continuous.

Theorem 3.1 (The Extreme Value Theorem)**.** *A continuous function on a closed bounded interval*

$$f : [a, b] \to \mathbb{R}$$

attains both a minimum and a maximum value.

Lemma: The image of a continuous function on a closed bounded interval is bounded above, that is, there is a number M such that

$$f(x) \leq M \qquad \forall x \in [a, b]$$

Theorem 3.2 (The Intermediate Value Theorem)**.** *Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous. Let c be a number strictly between $f(a)$ and $f(b)$; that is*

$$f(a) < c < f(b) \qquad \text{or} \qquad f(b) < c < f(a)$$

Then there is a point x_0 in the open interval (a, b) at which $f(x_0) = c$

Definition 3.2. A subset of D of $\mathbb{R}$ is said to be convex provided that whenever the points u an v are in D and $u < v$, then the whole interval $[u, v]$ is contained in D.

Theorem 3.3. *Let I be an interval and suppose that the function $f : I \to \mathbb{R}$ is continuous. Then its image $f(I)$ also is an interval.*

Definition 3.3. A function $f : D \to \mathbb{R}$ is said to be uniformly continuous provided that whenever $\{u_n\}$ and $\{v_n\}$ are sequences in D such that

$$\lim_{n\to\infty} [u_n - v_n] = 0$$

then

$$\lim_{n\to\infty} [f(u_n) - f(v_n)] = 0$$

Theorem 3.4. *A continuous function on a closed bounded interval,*

$$f : [a, b] \to \mathbb{R}$$

is uniformly continuous.

3.1 The $\epsilon - \delta$ Criterion for Continuity

Definition 3.4. A function $f : D \to \mathbb{R}$ is said to satisfy the $\epsilon - \delta$ criterion at a point x_0 in the domain D provided that for each positive number ϵ there is a positive number δ such that for x in D

$$|f(x) - f(x_0)| < \epsilon \qquad \text{if } |x - x_0| < \delta$$

This can be reworded as "For each symmetric band of width 2ϵ about the line $y = f(x_0)$ (no matter how small this width is), there is an interval $(x_0 - \delta, x_0 + \delta)$, centered at x_0 and of diameter $2\delta > 0$, such that the graph of the restriction of f to this interval lies in the given band."

Theorem 3.5. *For a function $f : D \to \mathbb{R}$ and a point x_0 in its domain D, the following two assertions are equivalent.*

1. *The function f is continuous at x_0; that is, for a sequence $\{x_n\}$ in D,*

$$\lim_{n\to\infty} f(x_n) = f(x_0) \qquad \textit{if } \lim_{n\to\infty} x_n = x_0$$

2. *The $\epsilon - \delta$ criterion at the point x_0 holds; that is, for each positive number ϵ there is a positive number δ such that for x in D,*

$$|f(x) - f(x_0)| < \epsilon \qquad \textit{if } |x - x_0| < \delta$$

Definition 3.5 (The $\epsilon - \delta$ Criterion on the Domain)**.** A function $f : D \to \mathbb{R}$ is said to satisfy the $\epsilon - \delta$ criterion on the domain D provided that for each positive number ϵ there is a positive number δ such that for all u, v in D,

$$|f(u) - f(v)| < \epsilon \qquad \text{if } |u - v| < \delta$$

Theorem 3.6. *For a function $f : D \to \mathbb{R}$ the following two assertions are equivalent:*

1. *The function is uniformly continuous; that is, for two sequences $\{u_n\}$ and $\{v_n\}$ in D,*

$$\lim_{n\to\infty} [f(u_n) - f(v_n)] = 0 \qquad \textit{if } \lim_{n\to\infty} [u_n - v_n] = 0$$

2. *The function f satisfies the $\epsilon - \delta$ criterion at the domain D; that is, for each positive number ϵ there is a positive number δ such that for u, v in D,*

$$|f(u) - f(v)| < \epsilon \qquad \textit{if } |u - v| < \delta$$

3.2 Images and Inverses; Monotone Functions

Definition 3.6. The function f is called monotonically increasing provided that

$$f(v) \geq f(u) \qquad \forall u, v \in D | v > u$$

Decreasing in the reverse. If a function is one or the other, it is said to be monotone. If the inequality is changed to be strict, then we call it strictly monotone.

Theorem 3.7. *Suppose that the function f is monotone. If its image $f(D)$ is an interval, then the function f is continuous.*

Corollary: Let I be an interval and suppose that the function $f : I \to \mathbb{R}$ is monotone. Then the function f is continuous iff its image $f(I)$ is an interval.

Theorem 3.8. *Let I be an interval and suppose that the function $f : I \to \mathbb{R}$ is strictly monotone. Then the inverse function $f^{-1} : f(I) \to \mathbb{R}$ is continuous.*

Definition 3.7. For $x > 0$ and rational number $r = m/n$, where m and n are integers with n positive, we define

$$x^r \equiv (x^m)^{(1/n)}$$

3.3 Limits

Definition 3.8. For a set D of real numbers, the number x_0 is called a limit point of $\mathbb{D}$ provided that there is a sequence of points in $D \setminus \{x_0\}$ that converges to x_0.

Definition 3.9. Given a function $f : D \to \mathbb{R}$ and a limit point x_0 of its domain D, for a number l, we write

$$\lim_{x \to x_0} f(x) = l$$

Theorem 3.9. *For functions f and g, and a limit point x_0 of their domains D, suppose that*

$$\lim_{x \to x_0} f(x) = A \qquad \text{and} \qquad \lim_{x \to x_0} g(x) = B$$

Then

$$\lim_{x \to x_0} [f(x) + g(x)] = A + B$$
$$\lim_{x \to x_0} [f(x)g(x)] = AB$$

and if $B \neq 0$ and $g(x) \neq 0$ for all x in D,

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{A}{B}$$

This extends to function compositions.

4 Differentiation

An open interval $I = (a, b)$ that contains the point x_0 is called a neighborhood of x_0.

Definition 4.1. Let I be a neighborhood of x_0. Then the function f is said to be differentiable at x_0 provided that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, in which case we denote this limit by $f'(x_0)$ and call it the derivative of f at x_0. If the function is differentiable at every point in I, we say that f is differentiable, and call f' the derivative of f.

A differentiable function is continuous.

Some rules:

1. $(x^n)' \Rightarrow nx^{n-1}$
2. $(f + g)' \Rightarrow f' + g'$
3. $(fg)' \Rightarrow f'g + fg'$
4. Polynomials are differentiable.
5. $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$

Theorem 4.1 (Derivative of the Inverse)**.** *Let I be a neighborhood of x_0 and let the function $f : I \to \mathbb{R}$ be strictly monotone and continuous. Suppose that f is differentiable at x_0 and that $f'(x_0) \neq 0$. Define $J = f(I)$. Then the inverse $f^{-1} : J \to \mathbb{R}$ is differentiable at the point $y_0 = f(x_0)$ and*

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}$$

Corollary: Let I be an open interval and suppose that the function $f : I \to \mathbb{R}$ is strictly monotone and differentiable with a nonzero derivative at each point in I. Define $J = f(I)$. Then the inverse function $f^{-1} : J \to \mathbb{R}$ is differentiable and

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))}$$

Theorem 4.2 (Rolle's Theorem)**.** *Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous and that the restriction of f to the open interval (a, b) is differentiable. Assume, moreover, that*

$$f(a) = f(b)$$

Then there is a point x_0 in the open interval (a, b) at which

$$f'(x_0) = 0$$

Theorem 4.3 (Mean Value Theorem)**.** *Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous and that the restriction of f to the open interval (a, b) is differentiable. Then there is a point x_0 in the open interval (a, b) at which*

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

Lemma: Let I be a neighborhood of x_0 and suppose that the function $f : I \to \mathbb{R}$ is differentiable at x_0. If the point x_0 is either a maximizer or a minimizer of the function $f : I \to \mathbb{R}$, then $f'(x_0) = 0$.

Theorem 4.4 (The Identity Criterion)**.** *A function f is said to be constant provided that there is some number c such that $f(x) = c$ for all x in D.*

This function is also constant if $f' = 0$ for all x in I.

Let functions g and h be differentiable. These functions differ by a constant[2] *iff $g'(x) = h'(x)$. These functions are the same if at some point x_0, $g(x_0) = h(x_0)$.*

Theorem 4.5 (The Monotone Criterion)**.** *If $f'(x) > 0$ for all x, then f is strictly increasing.*

Theorem 4.6 (The Maximizer and Minimizer Criterion)**.** *A point x_0 in the domain of a function f is said to be a local maximizer for f provided that there is some $\delta > 0$ such that*

$$f(x) \leq f(x_0) \qquad \text{for all } x \text{ in } D \text{ such that } |x - x_0| < \delta$$

This is flipped for minimizers.

If x_0 is a minimizer or maximizer, then $f'(x_0) = 0$. This also implies

$$\begin{cases} f'' > 0 \Rightarrow x_0 \text{ is minimizer} \\ f'' < 0 \Rightarrow x_0 \text{ is maximizer} \end{cases}$$

Theorem 4.7 (The Cauchy Mean Value Theorem)**.** *Suppose that the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous and that their restrictions to the open interval (a, b) are differentiable. Moreover assume that*

$$g'(x) \neq 0 \qquad \forall x \in (a, b)$$

Then there is a point x_0 in the open interval (a, b) at which

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}$$

Lemma: Let I be an open interval and n be a natural number and suppose that the function $f : I \to \mathbb{R}$ has n derivatives. Suppose also that at the point x_0 in I,

$$f^{(k)}(x_0) = 0 \qquad 0 \leq k \leq n - 1$$

Then, for each point $x \neq x_0$ in I, there is a point z strictly between x and x_0 at which

$$f(x) = \frac{f^{(n)}(z)}{n!}(x - x_0)^n$$

5 Differential Equations

We can provisionally solve a variety of differential equations.

[2] $g(x) = h(x) + c$

5.1 The Identity Criterion

A differentiable function $g : I \to \mathbb{R}$, where I is an open interval, is identically equal to 0 iff

1. its derivative $g' : I \to \mathbb{R}$ is identically equal to 0, and
2. there is some point $x_0 \in I$ at which $g(x_0) = 0$.

5.2 The Logarithmic Differential Equation

This is defined as

$$\begin{cases} F'(x) = \frac{1}{x} & \forall x > 0 \\ F(1) = 0 \end{cases}$$

Theorem 5.1. *Let the function $F : (0, \infty) \to \mathbb{R}$ satisfy the differential equation above. Then,*

1. $F(ab) = F(a) + F(b) \forall a, b > 0$.
2. $F(a^r) = rF(a)$ *if* $a > 0$ *and* r *is rational.*
3. *For each number c there is a unique positive number x such that $F(x) = c$.*

This function is the natural logarithm, denoted $\ln x$. Also, $ln 1 = e$, $a^x \equiv \exp(x \ln a)$

5.3 The Trigonometric Differential Equation

This is defined as

$$\begin{cases} f'' + f(x) = 0 & \forall x \\ f(0) = 1 \\ f'(0) = 0 \end{cases}$$

This is solved by sine and cosine.

6 Integration: Two Fundamental Theorems

6.1 Darboux Sums

Definition 6.1 (Partition)**.** Let a and b be real numbers with $a < b$. If n is a natural number and

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

then $P = \{x_n, \ldots, x_n\}$ is called a partition of the interval $[a, b]$. For each index $i \geq 0$, we call x_i a partition point of P, and if $i \geq 1$, we call the interval $[x_{i-1}, x_i]$ a partition interval of P.

Suppose that some function f is bounded, and the partition P is a partition of its domain. For each index i we define

$$m_i \equiv \inf \{f(x) | x \in [x_{i-1}, x_i]\}$$
$$M_i \equiv \sup \{f(x) | x \in [x_{i-1}, x_i]\}$$

We then define

$$L(f,P) \equiv \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$
$$U(f,P) \equiv \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

U is the upper Darboux Sum based on the partition P, and L is the lower. It follows from the definitions above that

$$m_i \le M_i$$

Therefore, for any partition P,

$$L(f,P) \le U(f,P)$$

Based on our intuitive definition of the integral, it also follows that

$$L(f,P) \le \int_a^b f \le U(f,P)$$

Theorem 6.1 (The Refinement Lemma)**.** *Suppose that the function $f : [a,b] \to \mathbb{R}$ is bounded and that P is a partition of its domain $[a,b]$. If P^* is a refinement of P, then*

$$L(f,P) \le L(f,P^*)$$
$$U(f,P^*) \le U(f,P)$$

Theorem 6.2. *Suppose that the function f is bounded, and that P_1 and P_2 are partitions of its domain $[a,b]$. Then*

$$L(f,P_1) \le U(f,P_2)$$

6.2 The Archimedes-Riemann Theorem

Definition 6.2. Suppose that some function f is bounded. Then we say that f is integrable on $[a,b]$ (or just integrable) if

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

Theorem 6.3 (The Archimedes-Riemann Theorem)**.** *Let f be a bounded function. Then f is integrable on $[a,b]$ iff there is a sequence $\{P_n\}$ of partitions of the interval $[a,b]$ such that*

$$\lim_{n\to\infty} [U(f,P_n) - L(f,P_n)] = 0$$

Moreover, for any sequence of partitions,

$$\lim_{n\to\infty} L(f,P_n) = \int_a^b f \quad \textit{and} \quad \lim_{n\to\infty} U(f,P_n) = \int_a^b f$$

Definition 6.3. Let the function $[a, b] : \mathbb{R} \to \mathbb{R}$ be bounded and for each natural number n, let P_n be a partition of its domain $[a, b]$. Then $\{P_n\}$ is said to be an Archimedean sequence of partitions for f on $[a, b]$ provided that

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0$$

Definition 6.4 (Regular Partitions)**.** For a natural number n, the partition $P = \{x_0, \ldots, x_n\}$ of the interval $[a, b]$ defined by

$$x_i = a + i\frac{b-a}{n} \qquad \forall 0 \le i \le n$$

is called the regular partition of $[a, b]$ into n partition intervals. It is characterized by the fact that all partition intervals are the same length, namely $(b - a)/n$.

Definition 6.5 (The Gap of a Partition)**.** For a partition $P = \{x_0, \ldots, x_n\}$ of the interval $[a, b]$, we define the gap of P, denoted by gap P, to be the length of the largest partition interval of P, that is,

$$\text{gap } P \equiv \max_{1\le i\le n} [x_i - x_{i-1}]$$

6.3 Linearity

Theorem 6.4 (Linearity of the Integral)**.** *Let the two functions f and g be integrable. Then, for any two numbers, α and β, the function $\alpha f + \beta g$ is integrable, and*

$$\int_a^b [\alpha f + \beta g] = \alpha \int_a^b f + \beta \int_a^b g$$

6.4 Continuity and Integrability

Theorem 6.5. *Two Theorems in this section:*

1. *A continuous function on a closed, bounded interval is integrable.*
2. *Suppose that the function f is bounded on the closed interval $[a, b]$ and is continuous on the open interval (a, b). Then f is integrable on $[a, b]$ and the value of the integral $\int_a^b f$ does not depend on the values of f at the endpoints of the interval.*

6.5 The First Fundamental Theorem: Integrating Derivatives

Theorem 6.6 (Integrating Derivatives)**.** *Let the function F be continuous on the closed interval $[a, b]$ and be differentiable on the open interval (a, b). Moreover, suppose that its derivative $F' : (a, b) \to \mathbb{R}$ is both continuous and bounded. Then*

$$\int_a^b F'(x)\, dx = F(b) - F(a)$$

6.6 The Second Fundamental Theorem: Differentiating Integrals

Theorem 6.7 (The Mean Value Theorem for Integrals). *Suppose that the function f is continuous, then there is a point $x_0 \in [a, b]$ at which*

$$\frac{1}{b-a}\int_a^b f = f(x_0).$$

Theorem 6.8 (Differentiating Integrals). *Suppose that the function f is continuous. Then*

$$\frac{d}{dx}\left[\int_a^x f\right] = f(x) \qquad \forall x \in (a, b)$$

7 Approximation by Taylor Polynomials

7.1 Taylor Polynomials

Definition 7.1. Let I be a neighborhood of the point x_0. Two functions f and g are said to have contact of order 0 at x_0 provided that $f(x_0) = g(x_0)$. For a natural number n, the functions f and g are said to have contact of order n at x_0 provided that f and g have n derivatives and

$$f^{(k)}(x_0) = g^{(k)}(x_0) \qquad 0 \leq k \leq n$$

Theorem 7.1 (Taylor Polynomial). *Let I be a neighborhood of the point x_0 and let n be a nonnegative integer. Suppose that the function f has n derivatives. Then there is a unique polynomial of degree at most n that has contact of order n with the function f at x_0. This polynomial is defined by the formula*

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

7.2 The Lagrange Remainder Theorem

Using the Cauchy Mean Value Theorem, we can define the Lagrange Remainder Theorem.

Theorem 7.2 (Lagrange Remainder Theorem). *Let I be a neighborhood of the point x_0 and let n be a non-negative integer. Suppose that the function f has $n+1$ derivatives. Then, for each point $x \neq x_0$ in i, there is a point c strictly between x and x_0 such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

7.3 The Convergence of Taylor Polynomials

For some sequence of numbers $\{a_k\}$ that is indexed by the nonnegative integers, we define

$$s_n = \sum_{k=0}^{n} a_k$$

and obtain a new sequence $\{s_n\}$ called the sequence of partial sums.

Let I be a neighborhood of the point x_0, and suppose that the function f has derivatives of all orders. The nth Taylor Polynomial for f at x_0 is defined by

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Using our partial sum notation, if x is a point in I at which

$$\lim_{n\to\infty} p_n(x) = f(x)$$

we write

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

However, this only holds if

$$\lim_{n\to\infty} [f(x) - p_n(x)] = 0$$

Theorem 7.3 (Useful Lemma)**.** *For any number c,*

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0$$

Theorem 7.4. *Let I be a neighborhood of the point x_0 and suppose that the function f has derivatives of all orders. Suppose also that there are positive numbers r and M such that the interval $[x_0 - r, x_0 + r]$ is contained in I and that for every natural number n and every point x in the aforementioned interval,*

$$\left| f^{(n)}(x) \right| \leq M^n$$

Then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \qquad \text{if } |x - x_0| \leq r$$

Theorem 7.5 (Corollary)**.**

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$
$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}$$

7.4 The Cauchy Integral Remainder Theorem

If I is a neighborhood of the point x_0 and the function f is differentiable, then, by the Mean Value Theorem, for each point x in I, there is a point c strictly between x and x_0 such that

$$f(x) = f(x_0) + f'(c)(x - x_0)$$

If we further assume that the derivative f' is continuous, then, by the first fundamental theorem,

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\, dt$$

Theorem 7.6 (The Cauchy Integral Remainder Formula)**.** *Let I be a neighborhood of the point x_0 and n be a natural number. Suppose that the function f has $n+1$ derivatives, and that $f^{(n+1)}$ is continuous. Then for each point $x \in I$,*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \, dt$$

Theorem 7.7 (The Ratio Lemma for Sequences)**.** *Suppose that $\{c_n\}$ is a sequence of nonzero numbers with the property that*

$$\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \ell$$

1. *If $\ell < 1$, then*

$$\lim_{n\to\infty} c_n = 0$$

2. *If $\ell > 1$, then the sequence is unbounded.*

7.5 The Weierstrass Approximation Theorem

Theorem 7.8 (The Weierstrass Approximation Theorem)**.** *Let I be a closed bounded interval and suppose that the function f is continuous. Then for each positive number ϵ, there is a polynomial p such that*

$$|f(x) - p(x)| < \epsilon \qquad \text{for all points } x \in I$$

8 Sequences and Series of Functions

8.1 Sequences and Series of Numbers

Definition 8.1 (The Cauchy Convergence Criterion for Sequences)**.** A sequence of numbers $\{a_n\}$ is said to be a Cauchy Sequence provided that for each positive number ϵ there is an index N such that

$$|a_n - a_m| < \epsilon \qquad \text{if } n \geq N \text{ and } m \geq N$$

1. Every convergent sequence if Cauchy.
2. Every Cauchy Sequence is bounded.

Theorem 8.1 (The Cauchy Convergence Criterion for Sequences)**.** *A sequence of numbers converges iff it is a Cauchy Sequence.*

Theorem 8.2 (Convergence Tests for Series)**.** *The following tests help prove convergence:*

1. *Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges. Then*

$$\lim_{n\to\infty} a_n = 0$$

2. *For a number f such that $|r| < 1$,*

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

3. *(The Comparison Test) Suppose that $\{a_k\}$ and $\{b_k\}$ are sequences of numbers such that for index k,*

$$0 \leq a_k \leq b_k$$

   (a) *The series $\sum_{k=0}^{\infty} a_k$ converges if the series $\sum_{k=0}^{\infty} b_k$ converges.*
   (b) *The series $\sum_{k=0}^{\infty} b_k$ diverges if the series $\sum_{k=0}^{\infty} a_k$ diverges.*

4. *(The Integral Test) Let $\{a_k\}$ be a sequence of nonnegative numbers and suppose that the function f is continuous and monotonically decreasing and has the property that*

$$f(k) = a_k \qquad \forall k$$

Then the series $\sum_{k=0}^{\infty} a_k$ is convergent iff the sequence of integrals

$$\left\{ \int_1^n f(x)\, dx \right\}$$

is bounded.

5. *(The p-Test) For a positive number p, the series*

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges iff $p > 1$.

6. *(The Alternating Series Test) Suppose that $\{a_k\}$ is a monotonically decreasing sequence of nonnegative numbers that converges to 0. Then the series*

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$$

converges.

7. *(The Cauchy Convergence Criterion for Series) The series $\sum_{k=1}^{\infty} a_k$ converges iff for each positive number ϵ there is an index N such that*

$$|a_{n+1} + \cdots + a_{n+k}| \leq \epsilon$$

8. *(The Absolute Convergence Test) An absolutely convergent series converges, that is, the series $\sum_{k=1}^{\infty} a_k$ converges if the series $\sum_{k=1}^{\infty} |a_k|$ converges.*

9. *For the series $\sum_{k=1}^{\infty} a_k$, suppose that there is a number r with $0 \leq r < 1$ and an index N such that*

$$|a_{n+1}| \leq r|a_n| \qquad \forall n \geq N$$

then the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

10. *(The Ratio Test for Series) For the series $\sum_{k=1}^{\infty} a_k$, suppose that*

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \rho$$

   (a) *If $\rho < 1$, the series converges absolutely.*

   (b) *If $\rho > 1$, the series diverges.*

8.2 Pointwise Convergence of Sequences of Functions

Definition 8.2. Given a function f and a sequence of function $\{f_n : D \to \mathbb{R}\}$, we say that the sequence converges pointwise to f, provided that for each point $x \in D$,

$$\lim_{n\to\infty} f_n(x) = f(x)$$

8.3 Uniform Convergence of Sequences of Functions

Definition 8.3. Given a function f and a sequence of functions $\{f_n\}$, the sequence is said to converge uniformly to f provided that for each positive number ϵ there is an index N such that,

$$|f(x) - f_n(x)| < \epsilon \qquad \forall n \geq N, x \in D$$

Definition 8.4. The sequence of functions is said to be uniformly Cauchy provided that for each positive number ϵ, there is an index N such that

$$|f_{n+k}(x) - f_n(x)| < \epsilon$$

for every index $n \geq N$, every natural number k, and every point $x \in D$.

Theorem 8.3 (The Weierstrass Uniform Convergence Criterion)**.** *The sequence of function $\{f_n\}$ converges uniformly to a function f iff the sequence is uniformly Cauchy.*

8.4 The Uniform Limit of Functions

Theorem 8.4 (Uniformly Convergent Sequences of Continuous Functions)**.** *Suppose that $\{f_n\}$ is a sequence of continuous functions that converges uniformly to the function f. Then the limit function f also is continuous.*

Theorem 8.5 (Uniformly Convergent Sequences of Integrable Functions)**.** *Suppose that $\{f_n\}$ is a sequence of integrable functions that converges uniformly to the function f. Then the limit function f also is integrable. Moreover,*

$$\lim_{n\to\infty} \left[\int_a^b f_n \right] = \int_a^b f$$

Theorem 8.6 (Uniformly Convergent Sequences of Differentiable Functions)**.** *Let I be an open interval. Suppose that $\{f_n\}$ is a sequence of continuously differentiable functions that has the following two properties:*

1. *The sequence* $\{f_n\}$ *converges pointwise on* I *to the function* f*, and*
2. *The derived sequence* $\{f_n'\}$ *converges uniformly on* I *to the function* g*.*

Then the function f *is continuously differentiable, and*

$$f'(x) = g(x) \qquad \forall x \in I$$

Theorem 8.7 (Uniformly Convergent Sequences of Differentiable Functions (2))**.** *Let* I *be an open interval. Suppose that* $\{f_n\}$ *is a sequence of continuously differentiable functions that has the following two properties:*

1. *The sequence converges pointwise on* I *to the function* f*, and*
2. *The derived sequence* $\{f_n'\}$ *is uniformly Cauchy on* I*.*

Then the function f *is continuously differentiable, and for each* $x \in I$

$$\lim_{n\to\infty} f_n'(x) = f'(x)$$