1 Complex Numbers and Elementary Functions

1.1 Properties

We define an imaginary number as

$$i^2 = -1$$

While a complex number is defined as

$$z = x + iy$$

The common functions $\Re$ and $\Im$ yield the real and imaginary parts of a complex number respectively.[1] We can also express complex numbers in polar coordinates.

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Using Euler's Identity,

$$\cos\theta + i\sin\theta = e^{i\theta}$$

the alternate form is defined as

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$
$$r = \sqrt{x^2+y^2} = |z|$$
$$\tan\theta = \frac{y}{x}$$

The complex conjugate is defined as

$$x - iy \equiv re^{-i\theta}$$

We can define some common equivalences.

- $\exp(2\pi i) = 1$
- $\exp(\pi i) = -1$
- $\exp\left(\frac{\pi i}{2}\right) = i$
- $\exp\left(\frac{3\pi i}{2}\right) = -i$
- $\exp(i\theta_1)\exp(i\theta_2) = \exp(i(\theta_1+\theta_2))$
- $\exp(i\theta)^m = \exp(im\theta)$
- $\exp(i\theta)^{1/n} = \exp\left(\frac{i\theta}{n}\right)$

Another neat trick is to let $z = 1/t$ to analyze behavior at ∞.

[1] Also denoted as Re and Im

1.2 Stereographic Projection

We can visualize complex numbers with a stereographic projection. Zero is located at the North Pole, and infinity at the South Pole.

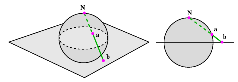

Figure 1: Stereographic Projection

These points are

$$X = \frac{4x}{|z|^2+4} \qquad Y = \frac{4y}{|z|^2+4} \qquad Z = \frac{2|z|^2}{|z|^2+4}$$

1.3 Elementary Functions

Similar to Real Analysis, we can define a neighborhood of some point z as the region enclosed by

$$|z - z_0| < \epsilon$$

As with sets, these can be closed, bounded, regions, domains, etc....

We can also define functions of complex numbers, and as with real valued numbers, they mostly work the same. The simplest function is the power function.

$$f(z) = z^n$$

Which can be extended to define Polynomials and rational functions (as the result of dividing a polynomial function with another).

Limits also work the same, even with Radii of Convergence, etc.

Projections and Mappings work intuitively.

1.4 Example - Rootfinding

Solve for all roots of the following equation: $z^4 + 2z = 0$.

$z(z^3+2) = 0$, so $z = 0$ or $z^3 = -2$, and then $r^3 = 2$, $e^{3i\theta} = e^{i\pi} \Rightarrow \theta = \pi/3 + 2\pi n/3$, $n = 0, 1, 2$. Thus, the roots are

$$z = 0, 2^{1/3}e^{i\pi/3}, 2^{1/3}e^{i\pi} = -2^{1/3}, 2^{1/3}e^{5i\pi/3}$$

1.5 Limits

Theorem 1 ($\epsilon - \delta$ Limit Definition). *A complex limit can be defined as*

$$\lim_{z\to z_0} f(z) = w_0$$

if for every sufficiently small $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon \qquad |z - z_0| < \delta$$

This is the traditional $\epsilon - \delta$ format that we're used to from real analysis.

Similarly, a function is said to be continuous if for all z,

$$\lim_{z\to z_0} f(z) = z_0$$

The traditional definitions of Uniform and Absolute convergence also apply.

Using these limit definitions we can define the concept of a derivative.

$$f'(z_0) = \lim_{\Delta z\to 0}\left(\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}\right) = \lim_{z\to z_0}\left(\frac{f(z) - f(z_0)}{z - z_0}\right)$$

1.6 Visualization

Is tricky. Wrote some code to rotate a complex function with static output supported. The hard part is you basically have a four-dimensional surface, since you have two input variables, the real and imaginary parts, and two output variables, the real and imaginary parts. The most straightforward way to visualize is to graph the output real and imaginary parts separately.

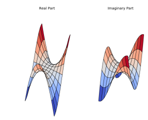

Figure 2: Visualization of $f(z) = z^3$

2 Analytic Functions and Integration

2.1 Analytic Functions

In order for a complex function to be differentiable, it has to satisfy the Cauchy-Riemann Conditions.

Theorem 2 (Cauchy-Riemann Conditions). *By writing the real and imaginary parts separately in the definition of a derivative, we get*

$$f(z) = u(x,y) + iv(x,y)$$
$$f'(z) = \lim_{\Delta x\to 0}\left(\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\frac{v(x+\Delta x,y) - v(x,y)}{\Delta x}\right)$$
$$= u_x(x,y) + iv_x(x,y)$$

Yielding the Cauchy-Riemann conditions,

$$u_x = v_y \qquad v_x = -u_y$$
$$u_r = \frac{v_\theta}{r} \qquad v_r = -\frac{u_\theta}{r}$$

Theorem 3. *The function $f(z) = u(x,y) + iv(x,y)$ is differentiable at a point $z = x + iy$ of a region in the complex plane if and only if the partial derivatives u_x, u_y, v_x, v_y, are continuous and satisfy the Cauchy-Riemann conditions at $z = x + iy$.*

3 2

[2] Holomorphic is sometimes used as well (or instead) of analytic.

[3] https://en.wikipedia.org/wiki/Analytic_function#Properties_of_analytic_functions

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2.5 Multivalued Functions

A simple example of this is the square root function which takes on different values for n even or odd.

$$z = w^2 \qquad w = \sqrt{z}$$
$$= r^{1/2}e^{i\theta_p/2}e^{n\pi i}$$

We can define these "points" where complex functions take on multiple values as branch points. In the same way that they're referred to as branch points, branches of a multivalued function are when we restrict to only one set of continuous values. A branch cut is this restriction process.[4]

Log is more complicated, and we define it as such.

$$w = \log(z) = \log r + i\theta_p + 2n\pi i, \qquad n = 0, \pm1, \pm2, \ldots, \qquad 0 \le \theta_p < 2\pi$$

2.6 Example - Branch Points/Cuts

Find the location of the branch points and discuss possible branch cuts for the following functions:

1. $(z - i)^{1/3}$
   Let $z - i = \epsilon e^{i\theta_p}$ which is a circular contour centered at $z = i$. We have just a power function in terms of $\zeta = z - i$, so $z = i$ and $z = \infty$ are branch points. Any line connecting $z = \infty$ and $z = i$ is a branch cut, e.g. $\{z = iy | y \in [1, +\infty)\}$ is as good as any. There are 3 distinct branches.
2. $\log\left(\frac{1}{z-2}\right)$
   $\log\left(\frac{1}{z-2}\right) = -\log(z - 2)$. Again, this is $-\log(z)$ but with shiftd origin. So the branch points are $z = 2$ and $z = \infty$. A branch cut must connect the branch points, it can be $\{z = x | x \in [2, +\infty)\}$ or $\{z = x | x \in (-\infty, 2]\}$.

2.7 Example - Rootfinding (cont.)

Solve for all values of z: $4 + 2e^{z+i} = 2$.

$$4 + 2e^{z+i} = 2 \Rightarrow e^{z+i} = -1 = e^{i\pi + 2\pi in}, n \in \mathbb{Z}$$

Therefore

$$z + i = i\pi + 2\pi in \Rightarrow z = i(\pi - 1 + 2\pi n), n \in \mathbb{Z}$$

[4] The real analogy here is a function like $\pm\sqrt{x}, x \in \mathbb{R}$. 0 is a branch point, and we often times just examine the branch where $\sqrt{x} > 0$. The analogous branch cut is $x > 0$.

2.8 Example - Branch Points/Cuts (cont.)

Find the location of the branch points and discuss a branch cut structure associated with the function:

- $f(z) = \frac{z-1}{z}$
  This is a rational function singular at $z = 0$, but single-valued, so no branch points.
- $f(z) = \log(z^2 - 3)$
  Here $z^2 - 3$ is entire single-valued function so the only branch points are those where $z^2 - 3 = 0$ or $z^2 - 3 = \infty$. Thus, there are three branch points, $z = \pm\sqrt{3}$, and $z = \infty$. A branch cut must make sure there is no possibility going around and single of them, in this case it must connect all three points. E.g. consider a cut on real axis $\{z = x | x \in [-3, +\infty)\}$.
- $f(z) = \exp\sqrt{z^2 - 1}$
  Since function e^z is entire (analytic on plane) the only possible branch points are those of $\sqrt{z^2-1}$, i.e. $z = \pm1$ and $z = \infty$. However, doing the circle argument $z - 1 = r_1e^{i\theta_1}$, $z + 1 = r_2e^{i\theta_2}$, $\theta_1 \to \theta_1 + 2\pi$, $\theta_2 \to \theta_2 + 2\pi$, one sets that $z = \infty$ is not a branch point since $\exp(2\pi i + 2\pi i)/2 = 1$ which corresponds to encircling both $z = 1$ and $z = -1$, equivalent to encircling just $z = \infty$. Thus, $z = \infty$ is not a branch point even for $\sqrt{z^2-1}$. But $z = \pm1$ are branch points, and a branch cut connecting them is $\{z = x | x \in [-1, 1]\}$.

2.9 More Complicated Multivalued Functions and Riemann Surfaces

If we have functions like the following

$$w = [(z - a)(z - b)]^{1/2}$$

We need to use a slightly more complicated branch cut/structure. We know that the points $z = a, b$ are both branch points (by letting $z = a + \epsilon_1 e^{i\theta_1}$ and as θ_1 varies from 0 to 2π, w jumps from $q^{1/2}$ to $-q^{1/2}$), and so we can define a branch cut as follows.

$$z - b = r_1e^{i\theta_1}$$
$$z - a = r_2e^{i\theta_2} \qquad 0 \le \theta_1, \theta_2 < 2\pi$$

Our equation now becomes

$$w = (r_1r_2)^{1/2}e^{i(\theta_1+\theta_2)/2}$$

This process extends to more complicated functions, as for any w of the form 2

$$w = [(z - x_1)(z - x_2)\cdots(z - x_n)]^m$$

we can define our branch cuts to be

$$z - x_k = r_ke^{i\theta_k}$$

yielding

$$w = (r_1r_2\cdots r_n)e^{mi(\theta_1+\theta_2+\cdots+\theta_n)}$$

2.10 Example - Branch Points/Cuts (cont.)

Find the location of branch points and discuss a branch cut structure associated with the function:

$$f(z) = \coth^{-1}\frac{z}{a} = \frac{1}{2}\log\left(\frac{z+a}{z-a}\right), a > 0$$

This is (up to a constant) log of rational function, so the branch points are those where $(z + a)/(z - a) = 0$ or ∞, i.e. $z = \pm a$. As for $z = \infty$, it is not a branch point, as the limit equals 1, not zero. A cut must connect the two points, so a possible one is interval $[-a, a]$ on the real axis.

2.11 Riemann Surfaces

Instead of considering the normal complex plane with arbitrary "cuts", it can be useful to instead consider a surface with multiple "sheets". Any multivalued function only has one point that corresponds to each point on the sheet. This way, for any given sheet, the function is single-valued.

For the function $w^{1/2}$, since we have two branches, our Riemann surface is two-sheeted. For the log function, since it is infinitely multivalued, we have infinite sheets.

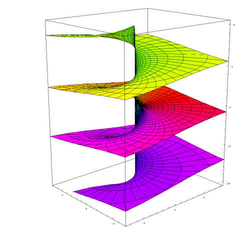

Figure 3: Riemann Surface for log(z)

2.12 Complex Integration

Consider a function $f(t) = u(t) + iv(t)$. This function is integrable if u and v are integrable (with the same properties applying).

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Defining a curve on the complex plane can be done parametrically, with form[5]

$$z(t) = x(t) + iy(t)$$

[5] These curves are

- **Simple Curve** or **Jordan Arc** if it does not intersect itself.
- **Simple Closed Curve** or **Jordan Curve** if the endpoints meet.

The path (contour) integral of function f on contour z is defined to be[6] 3

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt$$

This is really a line integral in the (x, y) plane.

Theorem 4. *Suppose $F(z)$ is an analytic function and that $f(z) = F'(z)$ is continuous in a domain D. Then for a contour C lying in D with endpoints z_1 and z_2*

$$\int_C f(z)\,dz = F(z_2) - F(z_1)$$

Since we can think of the parameterized complex plane as a vector field, for closed curves, we have

$$\oint_C f(z)\,dz = \oint_C F'(z)\,dz = 0$$

Note that everything here hinges on the analyticity of F and the continuity in domain D.

Theorem 5. *Let $f(z)$ be continuous on a contour C. Then*

$$\left|\int_C f(z)\,dz\right| \le ML$$

where L is the length of C and M is an upper bound for $|f|$ on C.

Arc length can be defined (from Calc III) for a parameterized curve with form $z(t) = u(t) + iv(t)$ as

$$\int_a^b \sqrt{(u'(t))^2 + (v'(t))^2}\,dt$$

Hey, this is nice and easy! If the given function is analytic on and in its domain, then it just equals zero! If there is a singularity on the inside of the domain, deform the contour so that you have 2 curves of opposite direction. Then $C_1 = C_2$, and you can just solve for the one that surrounds the singularity.

[6] Contours are defined as piecewise smooth connected arcs. Simple closed is referred to as a Jordan Contour

2.13 Example - Contour Integration

Evaluate $\int_C \overline{z}\,dz$ for a contour from $z = 0$ to $z = 1$ to $z = 1 + i$.

$$\int_C \overline{z}\,dz = \int_C (x - iy)(dx + i\,dy)$$
$$= \int_{x=0}^1 x\,dx + \int_{y=0}^1 (1 - iy)(i\,dy)$$
$$= \frac{1}{2} + i\left[y - iy^2/2\right]_0^1$$
$$= 1 + i$$

2.14 Cauchy's Theorem

Theorem 6 (Cauchy). *If a function f is analytic in a simply connected domain D, then along a simple closed contour C in D*

$$\oint_C f(z)\,dz = 0$$

We also require that $f'(z)$ is also continuous in D.

"If $f(z)$ is analytic everwhere interior to and on a simple closed contour C, then $\oint_C f(z)\,dz = 0$."

Again, NOTE that everything hinges on the fact that D must be simply connected. In order to use this, you need a simply connected domain D AND a simple closed contour C.

To best apply Cauchy's Theorem, we can use tricks like turning a complex contour into several simple contours, and deforming a simply connected domain so that the function is analytic on the domain.

2.15 Example - Cauchy's Theorem

Evaluate

$$\mathcal{I} = \frac{1}{2\pi i}\oint_C \frac{1}{(z-a)^m}\,dz, \qquad m = 1, 2, \ldots, M$$

where C is a simple closed contour.

The function $f(z) = 1/(z - a)^m$ is analytic for all $z \ne a$. Hence if C does not enclose $z = a$, then we have $\mathcal{I} = 0$. If C encloses $z = a$, we use Cauchy's Theorem to deform the contour to C_a, a small, but finite circle of radius r centered at $z = a$. Namely, 4

$$\int_C f(z)\,dz - \int_{C_a} f(z)\,dz = 0, \qquad f(z) = 1/(z-a)^m$$

We evaluate $\int_{C_a} f(z)\,dz$ by letting

$$z - a = re^{i\theta}, \qquad dz = ie^{i\theta}r d\theta$$

in which case

$$\mathcal{I} = \frac{1}{2\pi i}\oint_C \frac{1}{(z-a)^m}\,dz = \frac{1}{2\pi i}\int_0^{2\pi}\frac{1}{r^m e^{im\theta}}ie^{i\theta}r\,d\theta$$
$$= \frac{1}{2\pi i}\int_0^{2\pi} ie^{-i(m-1)\theta}r^{-m+1}\,d\theta = \delta_{m,1} = \begin{cases}1 & m = 1\\ 0 & \text{else}\end{cases}$$

Therefore,

$$\mathcal{I} = \begin{cases}0 & z = a \text{ outside } C\\ 0 & z = a \text{ inside } C, \quad m \ne 1\\ 1 & z = a \text{ inside } C, \quad m = 1\end{cases}$$

2.16 Example - Polynomials and Cauchy's Theorem

Let $P(z)$ be a polynomial of degree n, with n simple roots, none of which lie on a simple clsoed contour C. Evaluate

$$\mathcal{I} = \frac{1}{2\pi i}\oint_C \frac{P'(z)}{P(z)}\,dz$$

Because $P(z)$ is apolynomial with distinct roots, we can factor it as

$$P(z) = A(z - a_1)(z - a_2)\cdots(z - a_n)$$

Where A is the coefficient of the term of highest degree. Because

$$\frac{P'(z)}{P(z)} = \frac{d}{dz}(\log P(z)) \qquad = \frac{d}{dz}\log\left(A(z-a_1)(z-a_2)\cdots(z-a_n)\right)$$

it follows that

$$\frac{P'(z)}{P(z)} = \frac{1}{z-a_1} + \frac{1}{z-a_2} + \cdots + \frac{1}{z-a_1}$$

Hence, using the same method as above, we have

$$\mathcal{I} = \frac{1}{2\pi i}\oint_C \frac{P'(z)}{P(z)}\,dz = \text{number of roots lying within } C$$

2.17 Cauchy's Integral Formula, Its $\overline{\partial}$ Generalization and Consequences

Theorem 7. *Let $f(z)$ be analytic interior to and on a simple closed contour C. Then at any interior point z*

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

This is referred to as Cauchy's Integral Formula.

Theorem 8. *If $f(z)$ is analytic interior to and on a simple closed contour C then all the derivatives $f^{(k)}(z), k = 1, 2, \ldots$ exist in the domain D interior to C, and*

$$f^{(k)}(z) = \frac{k!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^{k+1}}\,d\zeta$$

Theorem 9. *All partial derivatives of u and v are continuous at any point where $f = u + iv$ is analytic.*

Theorem 10 (Lioville). *If $f(z)$ is entire and bounded in the z plane (including infinity), then $f(z)$ is a constant.*

Theorem 11 (Morera). *If $f(z)$ is continuous in a domain D and if*

$$\oint_C f(z)\,dz = 0$$

for every simple closed contour C lying in D, then $f(z)$ is analytic in D.

5

Theorem 12 (Maximum Principles). *1. If $f(z)$ is analytic in a domain D, then $|f(z)|$ cannot have a maximum in D unless $f(z)$ is a constant.*

2. If $f(z)$ is analytic in a bounded region D and $|f(z)|$ is continuous in the closed region $\overline{D}$, then $|f(z)|$ assumes its maximum on the boundary of the region.

Theorem 13 (Generalized Cauchy Formula). *If $\partial f/\partial\overline{\zeta}$ exists and is continuous in a region R bounded by a simple closed contour C, then at any interior point z*

$$f(z) = \frac{1}{2\pi i}\oint_C\left(\frac{f(\zeta)}{\zeta - z}\right)d\zeta - \frac{1}{\pi}\iint_R\left(\frac{\partial f/\partial\overline{\zeta}}{\zeta - z}\right)dA(\zeta)$$

so $\oint_C f(z)\,dz = -2\pi i/6 = -i\pi/3$.

Let $f(z)$ be an entire function with $|f(z)| \le C|z|$ for all z, where C is a constant. Show that $f(z) = Az$, where A is a constant.

Using the generalized Cauchy formula,

$$f'(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta$$

where $C = \{|\zeta - z| = R\}$ is the circle of radius R around z in the ζ-plane. Then

$$|f'| \le \frac{1}{2\pi}\oint_C \frac{|f(\zeta)|}{|\zeta - z|^2}|d\zeta| \le \frac{1}{2\pi}\int_0^{2\pi}\frac{C(|z| + R)}{R^2}Rd\theta = C(1 + |z|/R) = C$$

So $f'(z)$ is entire and bounded, so it is constant by Liouville theorem. Let $f'(z) = A$, then $f(z) = Az + B$, where A, B are constants. But, since $|f(z)| \le C|z|$ for all z, taking $|z| \to 0$, we get $B = 0$. Thus, $f(z) = Az$ as claimed.

2.19 Theoretical Developments

Theorem 14 (Cauchy-Goursat). *If a function $f(z)$ is analytic at all points interior to and on a simple closed contour, then*

$$\oint_C f(z)\,dz = 0$$

3 Sequences, Series, and Singularities of Complex Functions

3.1 Definitions of Complex Sequences, Series, and Their Basic Properties

We can denote a sequence of functions that converge to some given function as

$$\lim_{n\to\infty} f_n(z) = f(z) \Leftrightarrow |f_n(z) - f(z)| < \epsilon$$

If the limit does not exist, or is infinite, the sequence is said to diverge for those values of z.

6 We say the sequence of functions converges uniformly if we can choose N on only ϵ, and not z. In other words, if for any z, the n^{th} function is ϵ close to $f(z)$.

Theorem 15. *Let the sequence of functions $f_n(z)$ be continuous for each integer n and let $f_n(z)$ converge to $f(z)$ uniformly in a region $\mathcal{R}$. Then $f(z)$ is continuous, and for any finite contour C inside $\mathcal{R}$*

$$\lim_{n\to\infty}\int_C f_n(z)\,dz = \int_C f(z)\,dz$$

Theorem 16 (Weierstrass M Test). *Let $|b_j(z)| \le M_j$ in a region $\mathcal{R}$, with M_j constant. If $\sum_{j=1}^\infty M_j$ converges, then the series $S(z) = \sum_{j=1}^\infty b_j(z)$ converges uniformly in $\mathcal{R}$.*

Theorem 17 (Corollary: Ratio Test). *Suppose $|b_1(z)|$ is bounded, and*

$$\left|\frac{b_{j+1}(z)}{b_j(z)}\right| \le M < 1, \qquad j > 1$$

for M constant. Then the series

$$S(z) = \sum_{j=1}^\infty b_j(z)$$

is uniformly convergent.

3.2 Example - Convergence

Show that the following series converges uniformly in the given region: $\sum_{n=1}^\infty z^n, 0 \le |z| < R, R < 1$

$$\left|\sum_{n=1}^\infty z^n\right| \le \sum_{n=1}^\infty |z|^n \le \sum_{n=1}^\infty R^n = \frac{R}{1-R}$$

i.e. the series is bounded above by a convergent numerical series, which means numerical convergence by the Weierstrass M-test.

3.3 Example - Radius of Convergence

- z^{2n}

$$z^{2n} = (z^2)^n$$
$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{z^{2n}}{z^{2(n+1)}}\right| = \frac{1}{|z|^2}$$

Therefore it converges for $|z| < 1$ and the radius of convergence is $R = 1$.

- $n^n z^n$

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{n^n z^n}{(n+1)^{(n+1)} z^{n+1}}\right| = \frac{1}{(n+1)(1+1/n)^n |z|} = 0$$

Therefore $R = 0$.

3.4 Taylor Series

A power series about the point $z = z_0$ is defined as

$$f(z) = \sum_{j=0}^{\infty} b_j (z - z_0)^j$$
$$f(z + z_0) = \sum_{j=0}^{\infty} b_j z^j$$

With b_j, z_0 are constants. WLOG[7] we can work with

$$f(z) = \sum_{j=0}^{\infty} b_j z^j$$

which is the $z_0 = 0$ case.

Theorem 18. *If the series*

$$f(z) = \sum_{j=0}^{\infty} b_j z^j$$

converges for some $z_* \neq 0$, then it converges for all z in $|z| < |Z_*|$.

[7] Without Loss Of Generality

Morever, it converges uniformly in $|z| \leq R$ for $R < |Z_*|$.

Theorem 19 (Taylor Series). *Let $f(z)$ be analytic for $|z| \leq R$. Then*

$$f(z) = \sum_{j=0}^{\infty} b_j z^j$$

where

$$b_j = \frac{f^{(j)}(0)}{j!}$$

converges uniformly in $|z| \leq R_1 < R$.

The largest number R for which the power series converges inside the disk $|z| < R$ is called the radius of convergence.

Theorem 20. *Let $f(z)$ be analytic for $|z| \leq R$. Then the series obtained by differentiating the Taylor series termwise converges uniformly to $f'(z)$ in $|z| \leq R_1 < R$.*

Theorem 21. *If the power series converges for $|z| \leq R$, then it can be differentiated termwise to obtain a uniformly convergent series for $|z| \leq R_1 < R$.*

Theorem 22 (Comparison Test). *Let the series $\sum_{j=0}^{\infty} a_j z^j$ converge for $|z| < R$. If $|b_j| \leq |a_j|$ for $j \geq J$, then the series $\sum_{j=0}^{\infty} b_j z^j$ also converges for $|z| < R$.*

Theorem 23. *Let each of two functions $f(z)$ and $g(z)$ be analytic in a common domain D. If $f(z)$ and $g(z)$ coincide in some subportion $D' \subset D$ or on a curve Γ interior to D, then $f(z) = g(z)$ everywhere in D.*

where

$$C_n = \begin{cases} -1 & n \leq -1 \\ \frac{1}{2^{n+1}} & n \geq 0 \end{cases}$$

- Expand the function

$$f(z) = \frac{z}{(z-1)(z+2i)} \Rightarrow \frac{1/5 - 2i/5}{z-1} + \frac{4/5 + 2i/5}{z+2i}$$

in a Laurent series in the following regions.

  - $|z| < 1$

$$f(z) = \frac{1/5 - 2i/5}{z-1} + \frac{4/5 + 2i/5}{z+2i} = \left(\frac{2i-1}{5}\right)\sum_{n=0}^{\infty}\left(1 - \left(\frac{i}{2}\right)^n\right) z^n$$

  - $1 < |z| < 2$

$$f(z) = \frac{1/5 - 2i/5}{z-1} + \frac{4/5 + 2i/5}{z+2i} = \left(\frac{1-2i}{5}\right)\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \left(\frac{1-2i}{5}\right)\sum_{n=0}^{\infty}\left(\frac{i}{2}\right)^n z^n$$

  - $|z| > 2$

$$f(z) = \frac{1/5 - 2i/5}{z-1} + \frac{4/5 + 2i/5}{z+2i} = \left(\frac{1-2i}{5}\right)\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \left(\frac{4/5 + 2i/5}{z}\right)\sum_{n=0}^{\infty}\frac{(-2i)^n}{z^n}$$

3.9 Singularities of Complex Functions

An isolated singular point is a point where a given single-valued function is not analytic, but analytic in the neighborhood surrounding the point.

Removable singularities can be "removed" by using a Taylor or Laurent series expansion of the function.

An isolated singularity at z_0 of $f(z)$ is said to be a **pole** if $f(z)$ has the following representation.

$$f(z) = \frac{\phi(z)}{(z - z_0)^N}$$

Theorem 33. *Let $f(z) = N(z)/D(z)$ be a rational function such that the degree of $D(z)$ exceeds the degree of $N(z)$ by at least two. Then*

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$$

In other words, the integral converges.

Theorem 34 (Jordan's Lemma). *Suppose that on the circular arc C_R we have $f(z) \to 0$ uniformly as $R \to \infty$. Then*

$$\lim_{R\to\infty}\int_{C_R} e^{ikz} f(z)\,dz = 0 \qquad (k > 0)$$

We call this an Nth order pole if $N \geq 2$ and a simple pole if $N = 1$. The strength of the pole is $\phi(z_0)$.

An isolated singular point that is neither removable nor a pole is called an essential singular point. These have "full" Laurent series expansions.

Theorem 29. *If $f(z)$ has an essential singularity at $z = z_0$, then for any complex number w, $f(z)$ becomes arbitrarily close to w in a neighborhood of z_0. That is, given w, and any $\epsilon > 0$, $\delta > 0$, there is a z such that*

$$|f(z) - w| < \epsilon$$

whenever $0 < |z - z_0| < \delta$.

An entire function is one that is analytic everywhere on the complex plane. A meromorphic function is one that has only poles in the finite complex plane. A cluster point is an infinite sequence of isolated singular points that cluster in a neighborhood. A boundary jump discontinuity is where two analytic functions separated by a contour do not equal each other at the contour.

3.10 Example - Singularities

Discuss all singularities of the following functions.

- $\frac{z}{z^4+1}$. It is a rational function, it only has simple poles at the roots of

$$z^4 + 1 = 0, \qquad z = \left\{e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4}\right\}$$

- $\frac{\sin z}{z^3}$. Function $\sin z$ is entire, so

$$\frac{\sin z}{z^3} = \frac{z - z^3/3! - z^5/5! + \cdots}{z^3} = \frac{1}{z^2} - \frac{1}{6} + \frac{z^2}{120} + \cdots$$

so it has second order pole at $z = 0$ of strength 1, the only simple pole.

- $\frac{\cos z - 1}{z^2}$. The numerator is an entire function, so the only simple pole is $z = 0$.

$$\frac{\cos z - 1}{z^2} = \frac{1 - z^2/2! + z^4/4! - \cdots - 1}{z^2} \qquad = -\frac{1}{2} + \frac{z^2}{24} + \cdots$$

Figure 4: Small circular arc C_ϵ

4 1

Theorem 35. *1. Suppose that on the contour C_ϵ we have $(z - z_0)f(z) \to 0$ uniformly as $\epsilon \to 0$. Then*

$$\lim_{\epsilon\to 0}\int_{C_\epsilon} f(z)\,dz = 0$$

2. Suppose $f(z)$ has a simple pole at $z = z_0$ with residue $Res(f(z); z_0) = C_{-1}$. Then for the contour C_ϵ

$$\lim_{\epsilon\to 0}\int_{C_\epsilon} f(z)\,dz = i\phi C_{-1}$$

where the integration is carried out in the positive (counterclockwise) sense.

Theorem 24. *Let D_1 and D_2 be two disjoint domains, whose boundaries share a common contour Γ. Let $f(z)$ be analytic in D_1 and continuous in $D_1 \cup \Gamma$ and $g(z)$ be analytic in D_2 and continuous in $D_2 \cup \Gamma$, and let $f(z) = g(z)$ on Γ. Then the function*

$$H(z) = \begin{cases} f(z) & z \in D_1 \\ f(z) = g(z) & z \in \Gamma \\ g(z) & z \in D_2 \end{cases}$$

is analytic in $D = D_1 \cup \Gamma \cup D_2$. We say that $g(z)$ is the analytic continuation of $f(z)$.

Theorem 25. *If $f(z)$ is analytic and not identically zero in some domain D containing $z = z_0$, then its zeroes are isolated; that is, there is a neighborhood about $z = z_0$, $f(z_0) = 0$, in which $f(z)$ is nonzero.*

so $z = 0$ is a removable simple pole.

- $\coth z = \frac{\cosh z}{\sinh z}$, the ratio of two entire functions, so all simple poles are determined by $\sinh z = 0$, i.e. $z = i\pi k$ for all $k \in \mathbb{Z}$. Let $u = z - i\pi k$, then, since $\exp(i\pi k) = \exp(-i\pi k) = (-1)^k$, one has

$$\coth z = \frac{\cosh z}{\sinh z} = \frac{\cosh(u + i\pi k)}{\sinh(u + i\pi k)} = \frac{(-1)^k\cosh(u)}{(-1)^k\sinh(u)} = \frac{\cosh(u)}{\sinh(u)} = \frac{1}{u} + \frac{u}{3}$$

so all points $z = i\pi k$ are simple poles with residue 1.

3.11 Analytic Continuation

This is the process of extending the range of validity for a given function into a larger domain.

Theorem 30. *A function that is analytic in a domain D is uniquely determined either by values in some interior domain of D or along an arc interior to D.*

Theorem 31 (Monodromy Theorem). *Let D be a simply connected domain and $f(z)$ be analytic in some disk $D_0 \subset D$. If the function can be analytically continued along any two distinct smooth contours C_1 and C_2 to a point in D, and if there are no singular points enclosed within C_1 and C_2, then the result of each analytic continuation is the same and the function is single valued.*

Some functions can't be analytically continued due to a singularity referred to as a natural barrier.

3.12 Example - Analytic Continuation

Discuss the analytic continuation of the following function.

$$\sum_{n=0}^{\infty}\frac{z^{n+1}}{n+1} = \int_0^z\left(\sum_{n=0}^{\infty}u^n\right)du, \qquad |z| < 1$$

The first series indeed converges only for $|z| < 1$ where it defines an analytic function. For $|u| \leq |z| < 1$, the integrated series converges uniformly, so

$$\int_0^z\left(\sum_{n=0}^{\infty}u^n\right)du = \sum_{n=0}^{\infty}\int_0^z u^n\,du = \sum_{n=0}^{\infty}\frac{z^{n+1}}{n+1}$$

Theorem 36. *If on a circular arc C_R of radius R and center $z = 0$, $zf(z) \to 0$ uniformly as $R \to \infty$, then*

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$$

Theorem 37 (Argument Principle). *Let $f(z)$ be a meromorphic function defined inside and on a simple closed contour C, with no zeros or poles on C. Then*

$$I = \frac{1}{2\pi i}\oint_C\frac{f'(z)}{f(z)}\,dz = N - p = \frac{1}{2\pi}[\arg f(z)]_C$$

where N and P are the numbers of zeros and poles, respectively, of $f(z)$ inside C; where a multiple zero or pole is counted according to its multiplicity, and where $\arg f(z)$ is the argument of $f(z)$; that is, $f(z) = |f(z)|\exp(i\arg f(z))$ and $[\arg f(z)]_C$ denotes the change in the argument of $f(z)$ over C.

Theorem 38 (Rouché). *Let $f(z)$ and $g(z)$ be analytic on and inside a simple closed contour C. If $|f(z)| > |g(z)|$ on C, then $f(z)$ and $[f(z) + g(z)]$ have the same number of zeros inside the contour C.*

On the other hand, for $|u| \leq |z| < 1$,

$$\int_0^z\left(\sum_{n=0}^{\infty}u^n\right)du = \int_0^z\frac{du}{1-u} = \log(1-z)$$

where a branch analytic inside $|z| < 1$ is implied. Such a branh of $\log(1-z)$ is obtained e.g. if one makes a branch cut $[1, +\infty)$ on the positive real axis. Then the branch is analytic in $\mathbb{C} \setminus [1, +\infty)$, and, if one sets $\log 1 = 0$ to specify the branch, then it is equal to the first series in $|z| < 1$. Then this branch is the analytic continuation of the series to the region $\mathbb{C}$ minus the cut.

4 Residue Calculus and Applications of Contour Integration

4.1 Cauchy Residue Theorem

We've already discussed the Laurent expansion of $f(z)$ to be, for some analytic $f(z)$ in the region D, defined by $0 < |z - z_0| < \rho$, with $z = z_0$ isolated singular point,

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n$$
$$C_n = \frac{1}{2\pi i}\oint_C\frac{f(z)\,dz}{(z - z_0)^{n+1}}$$

where C is a simple closed contour in D. The negative part of the series is referred to as the principal part, while the coefficient C_{-1} is called the residue of $f(z)$ at z_0, denoted $C_{-1} = \text{Res}(f(z); z_0)$.

Theorem 32 (Cauchy Residue Theorem). *Let $f(z)$ be analytic inside and on a simple closed contour C, except for a finite number of isolated singular points $z_1, \ldots, z_N$ located inside C. Then*

$$\oint f(z)\,dz = 2\pi i\sum_{j=1}^{N} a_j$$

where a_j is the residue of $f(z)$ at $z = z_j$, denoted by $a_j = Res(f(z); z_j)$.

http://www.sosmath.com/trig/Trig5/trig5/trig5.html

Reciprocal identities

$$\sin u = \frac{1}{\csc u} \quad \cos u = \frac{1}{\sec u} \quad \tan u = \frac{1}{\cot u}$$
$$\csc u = \frac{1}{\sin u} \quad \sec u = \frac{1}{\cos u} \quad \cot u = \frac{1}{\tan u}$$

Pythagorean Identities

$$\sin^2 u + \cos^2 u = 1 \quad 1 + \tan^2 u = \sec^2 u \quad 1 + \cot^2 u = \csc^2 u$$

Quotient Identities

$$\tan u = \frac{\sin u}{\cos u} \quad \cot u = \frac{\cos u}{\sin u}$$

Theorem 26 (Laurent Series). *A function $f(z)$ analytic in an annulus $R_1 \leq |z - z_0| \leq R_2$ may be represented by the expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n$$

in the region $R_1 < R_a \leq |z - z_0| \leq R_b < R_2$, where

$$C_n = \frac{1}{2\pi i}\oint_C\frac{f(z)}{(z - z_0)^{n+1}}\,dz$$

and C is any simple closed contour in the region of analyticity enclosing the inner boundary $|z - z_0| = R_1$.

Theorem 27. *The Laurent series defined above of a function $f(z)$ that is analytic in an annulus $R_1 \leq |z - z_0| \leq R_2$ converges uniformly to $f(z)$ for $\rho_1 \leq |z - z_0| \leq \rho_2$, where $R_1 < \rho_1$ and $R_2 > \rho_2$.*

Theorem 28. *Suppose $f(z)$ is represented by a uniformly convergent series*

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n$$

in the annulus $R_1 \leq |z - z_0| \leq R_2$. Then $b_n = C_n$, with C_n previously defined.

This is a generic approach, however if $f(z)$ has a pole in the neighborhood of z_0, then it's a lot easier. Define

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}$$

where $\phi(z)$ is analytic in the neighborhood of z_0, m is a positive integer, and if $\phi(z_0) \neq 0$, f has a pole of order m. Then the residue of $f(z)$ at z_0 is given by

$$C_{-1} = \frac{1}{(m-1)!}\left(\frac{d^{m-1}}{dz^{m-1}}\phi\right)(z = z_0) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}((z - z_0)^m f(z))(z = z_0)$$

If it's the fraction of two rational functions, N and D, it can be as easy as $N(z_0)/D'(z_0)$. Sometimes we care about the residue at infinity.

$$\text{Res}(f(z); \infty) = \frac{1}{2\pi i}\oint_{C_\infty} f(z)\,dz = \frac{1}{2\pi i}\oint_{C_\epsilon}\left(\frac{1}{t^2}\right) f\left(\frac{1}{t}\right) dt$$

The value $w(z_j)$ is called the winding number of the curve C around the point z_j. This value represents the number of times that C winds around z_j. Positive means counterclockwise.

$$w(z_j) = \frac{1}{2\pi i}\oint_C\frac{dz}{z - z_j} = \frac{1}{2\pi i}[\log(z - z_j)]_C = \frac{\Delta\theta_j}{2\pi}$$

4.2 Example - Residues

Evaluate the integral

$$I = \frac{1}{2\pi i}\oint_C f(z)dz$$

where C is the unit circle centered at the origin, for the following $f(z)$.

- $f(z) = \frac{z+1}{z^3+a^3}, 0 < a < 1$ There are singular points at $z = -a, a\exp(i\pi/3)$, and $a\exp(-i\pi/3)$; all of these are inside the unit circle, so

$$\begin{aligned} I &= \text{Res}(f; -a) + \text{Res}(f; a\exp(i\pi/3)) + \text{Res}(f; a\exp(-i\pi/3)) \\ &= \frac{-a+1}{(-a - ae^{i\pi/3})(-a - ae^{-i\pi/33})} + \\ &\frac{ae^{i\pi/3}+1}{(ae^{i\pi/3} + a)(ae^{i\pi/3} - ae^{-i\pi/3})} + \\ &\frac{ae^{-i\pi/33}+1}{(ae^{-i\pi/3} + a)(ae^{-i\pi/3} - ae^{i\pi/3})} \\ &= 0 \end{aligned}$$

- $f(z) = \sin(1/z)$. Since $z = 0$ is the only singular point of $f(z)$, we do a Laurent series expansion about $z = 0$. Thus, $\text{Res}(f; 0) = 1$ and $I = 1$.

4.3 Evaluation of Certain Definite Integrals

We can use complex integration to solve real integrals as well.

4.3.1 Infinite Endpoints

For integrals of the form

$$I = \int_{-\infty}^{\infty} f(x)\,dx$$

where $f(x)$ is real valued. These integrals converge if the following two limits exist.

$$I = \lim_{L\to\infty}\int_{-L}^{\alpha} f(x)\,dx + \lim_{R\to\infty}\int_{\alpha}^{R} f(x)\,dx \qquad \alpha \text{ finite}$$

To evaluate this integral, we can take C to be a large semicircle that encloses all singularities of $f(z)$. Using this, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\sum_{j=1}^{N}\text{Res}(f(z); z_j)$$

Co-Function Identities

$$\sin\left(\frac{\pi}{2} - u\right) = \cos u \quad \cos\left(\frac{\pi}{2} - u\right) = \sin u \quad \tan\left(\frac{\pi}{2} - u\right) = \cot u$$
$$\csc\left(\frac{\pi}{2} - u\right) = \sec u \quad \sec\left(\frac{\pi}{2} - u\right) = \csc u \quad \cot\left(\frac{\pi}{2} - u\right) = \tan u$$

Even-Odd Identities

$$\sin(-u) = -\sin u \quad \cos(-u) = \cos u \quad \tan(-u) = -\tan u$$
$$\csc(-u) = -\csc u \quad \sec(-u) = \sec u \quad \cot(-u) = -\cot u$$

Sum-Difference Formulas

$$\sin(u \pm v) = \sin u\cos v \pm \cos u\sin v$$
$$\cos(u \pm v) = \cos u\cos v \mp \sin u\sin v$$
$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u\tan v}$$

Double Angle Formulas

$$\sin(2u) = 2\sin u\cos u$$
$$\cos(2u) = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u$$
$$\tan(2u) = \frac{2\tan u}{1 - \tan^2 u}$$

Power-Reducing/Half Angle Formulas

$$\sin^2 u = \frac{1 - \cos(2u)}{2}$$
$$\cos^2 u = \frac{1 + \cos(2u)}{2}$$
$$\tan^2 u = \frac{1 - \cos(2u)}{1 + \cos(2u)}$$

Sum-to-Product Formulas

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$
$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$
$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$
$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

Product-to-Sum Formulas

$$\sin u\sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$$
$$\cos u\cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$$
$$\sin u\cos v = \frac{1}{2}[\sin(u + v) + \sin(u - v)]$$
$$\cos u\sin v = \frac{1}{2}[\sin(u + v) - \sin(u - v)]$$