1 Separation of Variables

$$y' = 3t^2(1+y) \to \frac{dy}{dt} = 3t^2(1+y)$$
$$\frac{dy}{1+y} = 3t^2 \to \int \frac{dy}{1+y} = \int 3t^2$$
$$\ln|1+y| = t^3 + c \to |1+y| = e^c e^{t^3}$$
$$y = ce^{t^3} - 1,\ k \neq 0$$

2 Approximation Methods

2.1 Euler's Method (Tangent Line Method) - 1768

With a given function $y' = f(t,y)$ and a given set point p_0 we can approximate the line point by point.

$$\text{For the initial value problem } y' = f(t,y), y(t_0) = y_0$$
$$\text{Use the formulas } \begin{cases} t_{n+1} = t_n + h \\ y_{n+1} = y_n + hf(t_n, y_n) \end{cases} \tag{1}$$

2.1.1 Example

Obtain Euler approximation on $[0, 0.4]$ with step size 0.1 of

$$y' = -2ty + t \text{ and } y(0) = -1$$
$$h = 0.1, \begin{cases} t_0 = 0 \\ y_0 = -1 \end{cases}$$
$$\to \begin{cases} t_1 = t_0 + h = 0.1 \\ y_1 = y_0 + hf(t_0, y_0) = -1 \end{cases}$$
$$\to \begin{cases} t_2 = t_1 + h = 0.2 \\ y_2 = y_1 + hf(t_1, y_1) = -0.97 \end{cases}$$
$$\to \begin{cases} t_3 = t_2 + h = 0.3 \\ y_3 = y_2 + hf(t_2, y_2) = -0.9112 \end{cases}$$
$$\to \begin{cases} t_4 = t_3 + h = 0.4 \\ y_4 = y_3 + hf(t_3, y_3) = -0.826528 \end{cases}$$

2.2 Runge-Kutta Method of Approximation

If we have an IVP, we can calculate the next values with a process similar to (??)

$$\begin{cases} t_{n+1} = t_n + h \\ y_{n+1} = y_n + hk_{n2} \end{cases}$$
$$\text{Where}$$
$$k_{n1} = f(t_n, y_n)$$
$$k_{n2} = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_{n1}\right) \tag{2}$$

For more precision, we use the fourth order Runge-Kutta method. It is the most commonly used method both because of its speed as well as its relative precision.

$$\begin{cases} t_{n+1} = t_n + h \\ y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}) \end{cases}$$
$$\text{Where}$$
$$k_{n1} = f(t_n, y_n)$$
$$k_{n2} = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_{n1}\right) \tag{3}$$
$$k_{n3} = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_{n2}\right)$$
$$k_{n4} = f(t_n + h, y_n + hk_{n3})$$

3 Picard's Theorem

Theorem 1 (Picard's). *Suppose the function $f(t,y)$ is continuous on the region $R = \{(t,y) \mid a < t < b, c < y < d\}$ and $(t_0, y_0) \in R$. Then there exists a positive number h such that the IVP has a solution for t in the interval $(t_0 - h, t_0 + h)$. Furthermore, it $f_y(t,y)$ is also continuous on R, then that solution is unique.*

4 Linearity and Nonlinearity

An equation $F(x, x_2, x_3, \ldots, x_n) = c$ is linear if it is in the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$ where a_n are constants.

Furthermore, if $c = 0$, the equation is said to be homogeneous.

We can generalize the concept of a linear equation to a linear differential equation. A differential equation $F(y, y', y'', \ldots, y^n) = f(t)$ is linear if it is in the form: $a_n(t)\frac{d^ny}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1(t)\frac{d^1y}{dt^1} + a_0(t)\frac{d^0y}{dt^0} = f(t)$ where all function of t are assumed to be defined over some common interval I.

If $f(t) = 0$ over the interval I, the differential equation is said to be homogeneous.

We will also introduce some easier notation for linear algebraic equations: $\vec{\mathbf{x}} = [x_1, x_2, \ldots, x_n]$ and for linear differential equations: $\vec{\mathbf{y}} = [y^n, y^{n-1}, \ldots, y', y]$

We will also introduce the linear operator L: $L(\vec{\mathbf{x}}) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$

$$L(\vec{\mathbf{y}}) = a_n(t)\frac{d^ny}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1(t)\frac{d^1y}{dt^1} + a_0(t)\frac{d^0y}{dt^0}$$

4.1 Properties

A solution of the algebraic is any $\vec{\mathbf{x}}$ that satisfies the definition of linear algebraic equations, while a solution of the differential is for any $\vec{\mathbf{y}}$ that satisfies the definition of linear differential equations.

For homogeneous linear equations:

- A constant multiple of a solution is also a solution.
- The sum of two solutions is also a solution.

Linear Operator Properties:

- $L(k\vec{\mathbf{u}}) = kL(\vec{\mathbf{u}}), k \in \mathbb{R}$.
- $L(\vec{\mathbf{u}} + \vec{\mathbf{w}}) = L(\vec{\mathbf{u}}) + L(\vec{\mathbf{w}})$.

4.1.1 Superposition Principle

Let $\vec{\mathbf{u}}_1$ and $\vec{\mathbf{u}}_2$ be any solutions of the homogeneous linear equation $L(\vec{\mathbf{u}}) = 0$. Their sum is also a solution. A constant multiple is a solution for any constant k.

4.1.2 Nonhomogeneous Principle

Let $\vec{\mathbf{u}}_1$ be any solution to a linear nonhomogeneous equation $L(\vec{\mathbf{u}}) = c$ (algebraic) or $L(\vec{\mathbf{u}}) = f(t)$ (differential), then $\vec{\mathbf{u}} = \vec{\mathbf{u}}_n + \vec{\mathbf{u}}_p$ is also a solution, where $\vec{\mathbf{u}}$ is a solution to the associated homogeneous equation $L(\vec{\mathbf{u}}) = 0$.

4.2 Steps for Solving Nonhomogeneous Linear Equations

1. Find all $\vec{\mathbf{u}}_n$ of $L(\vec{\mathbf{u}}) = 0$.
2. Fina any $\vec{\mathbf{u}}_p of L(\vec{\mathbf{u}}) = f$.
3. Add them, $\vec{\mathbf{u}} = \vec{\mathbf{u}}_n + \vec{\mathbf{u}}_p$ to get all solutions of $L(\vec{\mathbf{u}}) = f$.

5 Solving 1^{st} Order Linear Differential Equations

5.1 Euler-Lagrange 2-Stage Method

To solve a linear differential equation in the form $y' + p(t)y = f(t)$ using this method:

1. Solve $y' + p(t)y = 0$ by separation of variables to get $y_n = ce^{-\int p(t)\,dt}$
2. Solve $v'(t)e^{-\int p(t)\,dt} = f(t)$ for $v(t)$ to get the particular solution $y_p = v(t)e^{-\int p(t)\,dt}$
3. Combine to get

$$y(t) = y_n + y_p = ce^{-\int p(t)\,dt} + e^{-\int p(t)\,dt}\int f(t)e^{\int p(t)\,dt}\,dt \tag{4}$$

5.2 Integrating Factor Method

1. Find the integrating factor $\mu(t) = e^{\int p(t)\,dt}$ (Note, $\int p(t)\,dt$ can be *any* antiderivative. In other words, don't bother with the addition of a constant.)
2. Multiply each side by the integrating factor to get $\mu(t)(y' + p(t)y) = f(t)\mu(t)$ Which will *always* reduce to $\frac{d}{dt}\left(e^{\int p(t)\,dt}y(t)\right) = f(t)e^{\int p(t)\,dt}$
3. Take the antiderivative of both sides $e^{\int p(t)\,dt}y(t) = \int f(t)e^{\int p(t)\,dt}dt + c$
4. Solve for y

$$y(t) = e^{\int p(t)\,dt}\int f(t)e^{\int p(t)\,dt}dt + ce^{\int p(t)\,dt} \tag{5}$$

5.2.1 Example

$$\frac{dy}{dt} - y = t$$
$$\mu(t) = e^{\int -1\,dt} \to e^{-t}$$
$$e^{-t}y = \int te^{-t}\,dt \to e^{-t}(-t-1) + c$$
$$y(t) = ce^t - t - 1$$

6 Applications of 1^{st} Order Linear Differential Equations

6.1 Growth and Decay

The function
$\frac{dy}{dt} = ky$
can be called the growth or decay equation depending on the sign of k. We can explicitly find the solution to these equations:

$$\text{For each } k\text{, the solution of the IVP}$$
$$\frac{dy}{dt} = ky, y(0) = y_0 \tag{6}$$
$$\text{Is given by}$$
$$y(t) = y_0e^{kt}$$

We can use these equations for a wide variety of different equations such as continuously compounding interest:

$$\frac{dA}{dt} = rA, A(0) = A_0 \tag{7}$$
$$A(t) = A_0e^{rt}$$

6.2 Mixing and Cooling

We can also use these models for mixing and cooling problems. A mixing problem consists of some amount of substance goes into a receptacle at a certain rate, and some amount of mixed substance comes out. We can model is as such.

$$\text{If } x(t) \text{ is the amount of dissolved substance, then}$$
$$\frac{dx}{dt} = \text{Rate In} - \text{Rate Out} \tag{8}$$
$$\text{Where } \begin{cases} \text{Rate In} = \text{Concentration in} \cdot \text{Flow Rate In} \\ \text{Rate Out} = \text{Concentration in} \cdot \text{Flow Rate Out} \end{cases}$$

We can also use these for cooling problems. Newton's law of cooling is as follows.

$$\frac{dT}{dt} = k(M - T) \tag{9}$$
$$\text{Where } \begin{cases} T \to \text{Temperature of the Object} \\ M \to \text{Temperature of the Surroundings} \end{cases}$$

7 Systems of Differential Equations

If one or more functions are dependent on other functions, then we call them

$$\text{Coupled} \begin{cases} y' = xy \\ x' = yx \end{cases}$$

coupled. Otherwise we call them decoupled.

$$\text{Decoupled} \begin{cases} y' = yt \\ x' = xt \end{cases}$$

7.1 Autonomous First Order System

Autonomous systems are not dependent on t, so we can treat them a little differently. For these equations we can use a phase plane, vector field, and the trajectory of the solution.

The functions $x(t)$ and $y(t)$ can give us a parametric curve. This means that at any given point on the curve, we also have a tangent vector given by $\frac{dy}{dt}$ and $\frac{dx}{dt}$.

Every solution of a system we call a state of the system, and the collection of all the trajectories and states is called a phase portrait.

An equilibrium point for this two dimensional system is an (x,y) point where $\frac{dy}{dt} = 0 = \frac{dx}{dt}$

7.2 Graphical Methods for Solving

Sketching is a pain in the ass...Therefore there are a couple tricks that we can use to make our lives easier.

We can use nullclines to more easily draw the solutions. Nullclines are an adaptation of previously mentioned isoclines (??). A V nullcline is an isocline of vertical slopes where $x' = 0$. An H nullcline is an isocline of horizontal slopes where $y' = 0$. Equilibria occurs at the point where these two nullclines intersect.

Note, when existence and uniqueness hold for an autonomous system, phase plane trajectories never cross.

7.3 Quick Sketching Outline for Phase Portraits

1. Nullclines and Equilibria
   - Where $x' = 0$, slopes are vertical.
   - Where $y' = 0$, slopes are horizontal.
   - Where $x' = y' = 0$, we have equilibria.
2. Left-Right Directions
   - Where x' is positive, arrows point right.
   - Where x' is negative, arrows point left.
3. Up-Down Directions
   - Where y' is positive, arrows point up.
   - Where y' is negative, arrows point down.
4. Check Uniqueness
   Where phase plane trajectories do not cross, we have uniqueness.

7.4 Applications of Systems of Differential Equations

7.4.1 Predator-Prey Assumptions

In the absence of foxes, the rabbit population will grow with the Malthusian Growth Law: $\frac{dR}{dt} = a_RR, a_R > 0$ In the absence of rabbits, the fox population will die off according to the law: $\frac{dF}{dtt} = -a_FF, a_F > 0$ When both foxes and rabbits are present, the number of interactions is $\propto$ the product of the population sizes, with inverse behavior. Thus we can get the Lotka-Volterra Equations for the predator prey model:

$$\begin{cases} \frac{dR}{dt} = a_RR - c_RRF \\ \frac{dF}{dt} = -a_FF - c_FRF \end{cases} \tag{10}$$

8 Matrices

8.1 Definitions

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_1 & b_2 & b_j & \cdots & b_n \\ c_1 & c_2 & c_3 & \cdots & c_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1 & m_2 & m_3 & \cdots & m_n \end{bmatrix} \tag{11}$$

We can also describe these matrices by saying it has order $m \times n$ where m and n are the row and column sizes respectively. Two matrices are equal if they have the same m and n and the values contained are equal. We can also have matrices with orders $m \times 1$ or $n \times 1$ which are called column and row vectors.

If all entries are 0, we call it a zero matrix; however if all entries but the diagonal are zero, this is called an diagonal matrix. These diagonal number are called diagonal elements. A special diagonal matrix is the identity matrix, which is formed when the diagonal elements are ones.

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & 1 & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{12}$$

8.2 Addition and Multiplication

Each new element in the matrix is a result of the dot product between the corresponding row and column matrices.

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{bmatrix}$$
$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1p} \\ B_{21} & B_{22} & \cdots & B_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mp} \end{bmatrix} \tag{13}$$
$$\mathbf{AB} = \begin{bmatrix} A_1 \cdot B_1 & A_2 \cdot B_1 & \cdots & A_3 \cdot B_1 \\ A_2 \cdot B_1 & A_2 \cdot B_2 & \cdots & A_2 \cdot B_3 \\ \vdots & \vdots & \ddots & \vdots \\ A_m \cdot B_1 & A_m \cdot B_2 & \cdots & A_m \cdot B_n \end{bmatrix}$$

8.3 Matrix Transposition

We can flip a matrix diagonally so that its columns become rows and its rows become columns. We call this the transpose of the matrix, written $\mathbf{A}^T$.

8.3.1 Properties

- $(\mathbf{A}^T)^T = \mathbf{A}$
- $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
- $(k\mathbf{A})^T = k\mathbf{A}^T$ for any scalar k.
- $(\mathbf{AB})^T = \mathbf{A}^T\mathbf{B}^T$

9 Matrices and Systems of Linear Equations

9.1 Augmented Matrix

An augmented matrix is where two different matrices are combined to form a new matrix.

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{cccc|c} A_{11} & A_{12} & \cdots & A_{1m} & b_1 \\ A_{21} & A_{22} & \cdots & A_{2m} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} & b_n \end{array}\right] \tag{14}$$

9.2 Elementary Row Operations

- Interchange row i and i $R_i^* = R_j, R_j^* = R_i$
- Multiply row i by a constant. $R_i^* = cR_i$
- Leaving j untouched, add to i a constant times j. $R_i^* = R_i + cR_j$

These are handy when dealing with matrices and trying to obtain Reduced Row Echelon Form (??).

9.3 Reduced Row Echelon Form

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right] \tag{15}$$

- 0 rows are at the bottom.
- Leftmost non-zero entry is 1, also called the pivot (or leading 1).
- Each pivot is further to the right than the one above.
- Each pivot is the only non-zero entry in its column.

A less complete process gives us row echelon form, which allows for nonzero entries are allowed above the pivot.

9.4 Gauss Jordan Reduction

1. Given a system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$
2. Form augmented matrix $[A|b]$
3. Transform to RREF (??) using elementary row operations.
4. The linear matrix formed by this process has the same solutions as the initial system, however it is much easier to solve.

9.5 Existence and Uniqueness

If the RREF has a row that looks like: $[0, 0, 0, \cdots, 0|k]$ where k is a non-zero constant, then the system has no solutions. We call this inconsistent.

If the system has one or more solutions, we call it consistent.

In order to be unique, the system needs to be consistent.

- If every column is a pivot, the there is only one solution (unique solution).
- Else If most columns are pivots, there are multiple solutions (possibly infinite).
- Else the system is inconsistent.

9.6 Superposition, Nonhomogeneous Principle, and RREF

For any nonhomogeneous linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$, we can write the solutions as: $\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p$ Where $\vec{\mathbf{x}}_h$ represents vectors in the set of homogeneous solutions, and $\vec{\mathbf{x}}_p$ is a particular solution to the original equation.

We can use RREF to find $\vec{\mathbf{x}}_p$, and then, using the same RREF with $\vec{\mathbf{b}}$ replaced by $\vec{\mathbf{0}}$, find $\vec{\mathbf{x}}_h$.

The rank of a matrix r equals the number of pivot columns in the RREF. If r equals the number of variables, there is a unique solution. Otherwise if there is less, then it is not unique.

9.7 Inverse of a Matrix

When given a system of equations like: $\begin{cases} x + y = 1 \\ 4x + 5y = 6 \end{cases}$ we can rewrite it in the form: $\begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ For this sort of matrix, we can find the inverse which is defined as the matrix that, when multiplied with the original, equals an Identity Matrix. In other words: $A^{-1}A = AA^{-1} = I$

9.7.1 Properties

- $(A^{-1})^{-1} = A$
- A and B are invertible matrices of the same order if $(AB) = A^{-1}B^{-1}$
- If A is invertible, then so is A^T and $(A^{-1})^T = (A^T)^{-1}$

9.7.2 Inverse Matrix by RREF

For an $n \times n$ matrix A, the following procedure either produces A^{-1}, or proves that it's impossible.

1. Form the $n \times 2n$ matrix $M = [A|I]$
2. Transform M into its RREF, R.
3. If the first n columns produce an Identity Matrix, then the last n are its inverse. Otherwise A is not invertible.

9.8 Invertibility and Solutions

The matrix vector equation $A\mathbf{x} = b$ where A is an $n \times n$ matrix has:

- A unique solution $x = A^{-1}b$ if and only if A is invertible.
- Either no solutions or infinitely many solutions if A is not invertible.

For the homogeneous equation $A\mathbf{x} = 0$, there is always one solution, $x = 0$ called the trivial solution.

Let $\mathbf{A}$ be an $n \times n$ matrix. The following statements apply.

- $\mathbf{A}$ is an invertible matrix.
- $\mathbf{A}^T$ is an invertible matrix.
- $\mathbf{A}$ is row equivalent to I_n.
- $\mathbf{A}$ has n pivot columns.
- The equation $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has only the trivial solution, $\vec{\mathbf{x}} = \vec{\mathbf{0}}$.
- The equation $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has a unique solution for every $\vec{\mathbf{b}}$ in $\mathbb{R}^n$.

9.9 Determinants and Cramer's Rule

9.9.1 2×2 Matrix

To find the determinant of a 2×2 matrix, the determinant is the diagonal products subtracted. This process is demonstrated below.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{16}$$
$$|A| = a_{22} \cdot a_{11} - a_{12} \cdot a_{21}$$

9.9.2 Definitions

Every element of a $n \times n$ matrix has an associated minor and cofactor.

- Minor $\to$ A $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of A.
- Cofactor $\to$ The scalar $C_{ij} = (C-1)^{i+j}|M_{ij}|$

9.9.3 Recursive Method of an $n \times n$ matrix A

We can now determine a recursive method for any $n \times n$ matrix.

Using the definitions declared above, we use the recursive method that follows.

$$|A| = \sum_{j=1}^{n} a_{ij}C_{ij} \tag{17}$$

Find j and then finish with the rules for the 2×2 matrix defined above in (??).

9.9.4 Row Operations and Determinants

Let A be square.

- If two rows of A are exchanged to get B, then $|B| = -|A|$.
- If one row of A is multiplied by a constant c, and then added to another row to get B, then $|A| = |B|$.
- If one row of A is multiplied by a constant c, then $|B| = c|A|$.
- If $|A| = 0$, A is called singular.

For an $n \times n$ A and B, the determinant $|AB|$ is given by $|A||B|$.

9.9.5 Properties of Determinants

- If two rows of $\mathbf{A}$ are interchanged to equal $\mathbf{B}$, then $|\mathbf{B}| = -|\mathbf{A}|$
- If one row of $\mathbf{A}$ is multiplied by a constant k, and then added to another row to produce matrix $\mathbf{B}$, then $|\mathbf{B}| = |\mathbf{A}|$
- If one row of $\mathbf{A}$ is multiplied by k to produce matrix $\mathbf{B}$, then $|\mathbf{B}| = k|\mathbf{A}|$
- If $|AB| = 0$, then either $|A|$ or $|B|$ must be zero.

- $|A^T| = A$
- If $|\mathbf{A}| \neq 0$, then $|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}$.
- If A is an upper or lower triangle matrix[1], then the determinant is the product of the diagonals.
- If one row or column consists of only zeros, then $|A| = 0$.
- If two rows or columns are equal, then $|A| = 0$.
- A is invertible.
- A^T is also invertible.
- A has n pivot columns.
- $|A| \neq 0$
- If $|A| = 0$ it is called singular, otherwise it is nonsingular.

9.9.6 Cramer's Rule

For the $n \times n$ matrix A with $|A| \neq 0$, denote by A_i the matrix obtained from A by replacing its ith column with the column vector $\mathbf{b}$. Then the ith component of the solution of the system is given by:

$$x_i = \frac{|A_i|}{|A|} \tag{18}$$

10 Vector Spaces and Subspaces

A vector space $\mathcal{V}$ is a non-empty collection of elements that we call vectors, for which we can define the operation of vector addition and scalar multiplication:

1. Addition: $\vec{\mathbf{x}} + \vec{\mathbf{y}}$
2. Scalars: $c\vec{\mathbf{x}}$ where c is a constant.

that satisfy the following properties:

[1] A triangle matrix is one where either the lower or upper half is zero, e.g. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

1. $\vec{\mathbf{x}} + \vec{\mathbf{y}} \in \mathcal{V}$
2. $c\vec{\mathbf{x}} \in \mathcal{V}$

which can be condensed into a single equation: $c\vec{\mathbf{x}} + d\vec{\mathbf{y}} \in \mathcal{V}$ which is called closure under linear combinations.

10.1 Properties

We have the properties from before, as well as new ones.

1. $\vec{\mathbf{x}} + \vec{\mathbf{y}} \in \mathcal{V} \leftarrow$ Addition
2. $c\vec{\mathbf{x}} \in \mathcal{V} \leftarrow$ Scalar Multiplication
3. $\vec{\mathbf{x}} + \vec{\mathbf{0}} = \vec{\mathbf{x}} \leftarrow$ Zero Element
4. $\vec{\mathbf{x}} + (-\vec{\mathbf{x}}) = (-\vec{\mathbf{x}}) + \vec{\mathbf{x}} = \vec{\mathbf{0}} \leftarrow$ Additive Inverse
5. $(\vec{\mathbf{x}} + \vec{\mathbf{y}}) + \vec{\mathbf{z}} = \vec{\mathbf{x}} + (\vec{\mathbf{y}} + \vec{\mathbf{z}}) \leftarrow$ Associative Property
6. $\vec{\mathbf{x}} + \vec{\mathbf{y}} = \vec{\mathbf{y}} + \vec{\mathbf{x}} \leftarrow$ Commutativity
7. $1 \cdot \vec{\mathbf{x}} = \vec{\mathbf{x}} \leftarrow$ Identity
8. $c(\vec{\mathbf{x}} + \vec{\mathbf{y}}) = c\vec{\mathbf{x}} + c\vec{\mathbf{y}} \leftarrow$ Distributive Property
9. $(c+d)\vec{\mathbf{x}} = c\vec{\mathbf{x}} + d\vec{\mathbf{x}} \leftarrow$ Distributive Property
10. $c(d\vec{\mathbf{x}}) = (cd)\vec{\mathbf{x}} \leftarrow$ Associativity

10.2 Vector Function Space

A vector function space is just a unique vector space where the elements of the space are functions.

Note, the solutions to linear and homogeneous differential equations form vector spaces.

10.2.1 Closure under Linear Combination

$$c\vec{\mathbf{x}} + d\vec{\mathbf{y}} \in \mathbb{V} \text{ whenever } \vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbb{V} \text{ and } c, d \in \mathbb{R} \tag{19}$$

10.2.2 Prominent Vector Function Spaces

- $\mathbb{R}^2 \to$ The space of all ordered pairs.
- $\mathbb{R}^3 \to$ The space of all ordered triples.
- $\mathbb{R}^n \to$ The space of all ordered n-tuples.
- $\mathbb{P} \to$ The space of all polynomials.
- $\mathbb{P}_n \to$ The space of all polynomials with degree $\leq n$.
- $\mathbb{M}_{mn} \to$ The space of all $m \times n$ matrices.
- $\mathbb{C}(I) \to$ The space of all continuous functions on the interval I (open, closed, finite, and infinite).
- $\mathbb{C}^n(I) \to$ Same as above, except with n continuous derivatives.
- $\mathbb{C}^n \to$ The space of all ordered n-tuples of complex numbers.

10.3 Vector Subspaces

Theorem: A non-empty subset $\mathbb{W}$ of a vector space $\mathbb{V}$ is a subspace of $\mathbb{V}$ if it is closed under addition and scalar multiplication:

- If $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in \mathbb{W}$, than $\vec{\mathbf{u}} + \vec{\mathbf{V}} \in \mathbb{W}$.
- If $\vec{\mathbf{u}} \in \mathbb{W}$ and $c \in \mathbb{R}$, than $c\vec{\mathbf{u}} \in \mathbb{W}$.

We can rewrite this to be more efficient:

$$\text{If } \vec{\mathbf{u}}, \vec{\mathbf{v}} \in \mathbb{W} \text{ and } a, b \in \mathbb{R}, \text{ than } a\vec{\mathbf{u}} + b\vec{\mathbf{v}} \in \mathbb{W}. \tag{20}$$

Note, vector space does not imply subspace. All subspaces are vector spaces, but not all vector spaces are subspaces.

To determine if it is a subspace, we check for closure with the above theorem.

There are only a couple subspaces for $\mathbb{R}^2$:

- The zero subspace $\{(0,0)\}$.
- Lines passing through the origin.
- $\mathbb{R}^2$ itself.

We can call the zero and the set $\mathbb{V}$ themselves trivial subspaces, calling the subspace of lines passing through the origin the only non-trivial subspace in $\mathbb{R}^2$.

We can classify $\mathbb{R}^3$ similarly:

- Trivial:
  - Zero subspace
  - $\mathbb{R}^3$

  Non-Trivial
  - Lines that contain the origin.
  - Places that contain the origin.

10.3.1 Examples

- The set of all even functions.
- The set of all solutions to $y''' - y''t + y = 0$.
- $\{P \in \mathbb{P}; P(2) = P(3)\}$

11 Span, Basis and Dimension

11.1 Span

The span of a set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n\}$ of vectors in a vector space $\mathbb{V}$, denoted by Span$\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n\}$ is the set of all linear combinations of the vectors.

11.1.1 Example

For example, If $\vec{\mathbf{u}} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

Then we can write their span as

$$a\vec{\mathbf{u}} + b\vec{\mathbf{v}} = a\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3a \\ 2a + 2b \\ 2b \end{bmatrix}$$

11.2 Spanning Sets in $\mathbb{R}^n$

A vector $\vec{\mathbf{b}}$ in $\mathbb{R}^n$ is in Span$\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n\}$ where $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n\}$ are vectors in $\mathbb{R}^n$, provided that there is at least one solution of the matrix-vector equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, where A is the matrix whose column vectors are $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n\}$.

11.3 Span Theorem

For a set of vectors $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n\}$ in vector space $\mathbb{V}$, Span$\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n\}$ is subspace of $\mathbb{V}$.

11.4 Column Space

For any $m \times n$ matrix A, the column space, denoted Col A, is the span of the column vectors of A, and is a subspace of $|mathbbR^n$.

11.5 Linear Independence

A set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n\}$ of vectors in vector space $\mathbb{V}$ is linearly independent if no vector of the set can be written as a linear combination of the others. Otherwise it is linearly dependent.

This notion of linear independence also carries over to function spaces. A set of vector functions $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n\}$ in a vector space $\mathbb{V}$ is linearly independent on an interval I if for all t in I the only solution of $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_n\vec{\mathbf{v}}_n = \vec{\mathbf{0}}$ for $(c_1, c_2, \ldots, c_n \in \mathbb{R})$ is $c_i = 0$ for all i.

If for any value t_0 of t there is any solution with $c_i \neq \vec{\mathbf{0}}$, the vector functions are linearly dependent.

11.5.1 Testing for Linear Independence

1. (a) Put the system in matrix-vector form:
   $$\begin{bmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 & \cdots & \vec{\mathbf{v}}_n \\ \downarrow & \downarrow & \cdots & \downarrow \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{\mathbf{0}}$$
   (b) Analyze Matrix:
   The column vectors of A are linearly independent if and only if the solution $\vec{\mathbf{x}} = \vec{\mathbf{0}}$ is unique, which means $c_i = 0$ for all i.
   Any of the following also satisfy this condition for a unique solution:

- A is invertible.
- A has n pivot columns.
- $|A| \neq 0$

2. Suppose we have a set of vectors $\vec{\mathbf{v}}$. $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n\} \in \mathbb{R}^n, \dim(\vec{\mathbf{v}}) = m$ Then the set $\vec{\mathbf{v}}$ is linearly dependent if $n > m$ where n is the number of elements in $\vec{\mathbf{v}}$. *Note, this cannot prove the opposite. It only goes one way.* $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix} \right\}$ Is dependent

3. Columns of A are linearly independent if and only if $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has only the trivial solutions of n.

11.5.2 Linear Independence of Functions

One way to check a set of functions is to consider them as a one dimensional vector. $\vec{\mathbf{v}}_i(t) = f_n(t)$ Another method is the Wronskian:

To find the Wronskian of functions $f_1, f_2, \dots, f_n$ on I :

$$W[f_1, f_2, \dots, f_n] = \begin{bmatrix} f_1 & f_2 & \cdots & r_n \\ f_1' & f_2' & \cdots & r_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{n-1} & f_2^{n-1} & \cdots & r_n^{n-1} \end{bmatrix} \tag{21}$$

If $W \neq 0$ for all t on the interval I, where $f_1, f_2, \dots, f_n$ are defined, then the function space is a linearly independent set of functions on I.

11.6 Basis of a Vector Space

The set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n\}$ is a basis for vector space $\mathbb{V}$ provided that

- $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n\}$ is linearly independent.
- $\text{Span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n\} = \mathbb{V}$

Case One $\Delta > 0$	Real Unequal Roots $r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Overdamped Motion $y_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
Case Two $\Delta = 0$	Real Repeated Root $r = -\frac{b}{2a}$	Critically Damped Motion $y_h(t) = c_1 e^{rt} + c_2 t e^{rt}$
Case Three $\Delta < 0$	Complex Conjugate Roots $r_1, r_2 = \alpha \pm \beta i$, $\alpha = -\frac{b}{2a}, \beta = \frac{\sqrt{4ac - b^2}}{2a}$	Underdamped Motion $y_h(t) = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$

Table 1: Roots for Second Order Differential Equations in Characteristic Equation Form

12.4 Linear Independence

The Solution Space Theorem (**??**) provides us with the number of solutions in a bases for an nth order homogeneous differential equation (n).

- Starting with m solutions for the nth order case, if $m > n$ the solutions can be independent.
- If $m = n$, we must test using the concepts from before.
- If $m < n$, the set does not span the space.

12.4.1 Wronskian

The Wronskian also tells us about the linear independence of a set of functions. This Wronskian is identical to the Wronskian previously defined (**??**).

Suppose $\{y_1, y_2, \dots, y_n\}$ is a set of solutions of an nth order homogeneous differential equation.

$L(y) = a_n(t)y^n + a_{n-1}(t)y^{n-1} + \dots + a_1(t)y' + a_0(t)y = 0$

1. If $W[y_1, y_2, \dots, y_n] \neq 0$ at any point on (a, b), then the set is linearly independent.
2. If $W[y_1, y_2, \dots, y_n] = 0$ at every point on (a, b), then the set is linearly dependent.

12.5 Undetermined Coefficients

Let's assume $L(y) = a_n(t)y^n + a_{n-1}(t)y^{n-1} + \dots + a_1(t)y' + a_0(t)y = 0$ where $t \in$ some interval I.

- Equilibrium arrives at origin (Symmetric)
- Speed is determined by magnitude of the eigenvalues.

14.2 Linear Systems with Real Eigenvalues

To solve a system in the form

$\vec{\mathbf{x}} = A\vec{\mathbf{x}}$

1. Find eigenvalues of A.
2. Find associated eigenvectors.
3. Solution is in the form (for a 2×2 matrix at least) our solution is in the form: $\vec{\mathbf{x}}(t) = c_1 e^{\lambda_1 t}\vec{\mathbf{v}}_1 + c_2 e^{\lambda_2 t}\vec{\mathbf{v}}_2$

If there are insufficient eigenvalues (repeated eigenvalues), follow the method below.

1. Find the one eigenvalue.
2. Find its eigenvector.
3. Find $\vec{\mathbf{v}}$ such that $(A - \lambda I)\vec{\mathbf{u}} = \vec{\mathbf{v}}$.
4. Solution: $\vec{\mathbf{x}}(t) = c_1 e^{\lambda t}\vec{\mathbf{v}} + c_2 e^{\lambda t}(t\vec{\mathbf{v}} + \vec{\mathbf{u}})$.

14.3 Non-Real Eigenvalues

If we have a matrix A with non-real eigenvalues $\lambda_1, \lambda_2 = \alpha \pm i\beta$, the corresponding eigenvectors are also complex conjugate pairs in the form:

$\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2 = \vec{\mathbf{p}} \pm i\vec{\mathbf{q}}$

To solve:

1. For the first eigenvalue, find its eigenvector. The second eigenvector is a pair of the first.
2. Construct the real and non-real parts: $\begin{cases} \vec{\mathbf{x}}_r = e^{\alpha t}(\cos(\beta t)\vec{\mathbf{p}} - \sin(\beta t)\vec{\mathbf{q}}) \\ \vec{\mathbf{x}}_i = e^{\alpha t}(\sin(\beta t)\vec{\mathbf{p}} + \cos(\beta t)\vec{\mathbf{q}}) \end{cases}$
3. The general solution is defined as $\vec{\mathbf{x}}(t) = c_1\vec{\mathbf{x}}_r(t) + c_2\vec{\mathbf{x}}_i(t)$

11.6.1 Standard Basis for $\mathbb{R}^n$

$$\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \dots, \vec{\mathbf{e}}_n\}$$

where

$$\vec{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \vec{\mathbf{e}}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{22}$$

are the column vectors of the identity matrix I_n.

11.6.2 Example

A vector space can have different bases.

The standard basis for $\mathbb{R}^n$ is: $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2\}$ for $\vec{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{\mathbf{e}}_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ giving $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

But another basis for $\mathbb{R}^2$ is given by:

$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

11.7 Dimension of the Column Space of a Matrix

Essentially, the number of vectors in a basis.

11.7.1 Properties

- The pivot columns of a matrix A form a basis for Column A.
- The dimension of the column space, called the rank of A, is the number of pivot columns in A. rank $A = \dim(\text{Col}(A))$

11.7.2 Invertible Matrix Characterizations

Let A be an $n \times n$ matrix. The following are true.

- A is invertible.
- The column vector of A is linearly independent.
- Every column of A is a pivot column.
- The column vectors of A form a basis for $\text{Col}(A)$.
- Rank $A = n$

10 If $y_i(t)$ is a solution of $L(y) = f_i(t)$, then $y(t) = c_1 y_1(t) + c_2 y_2(t) + \dots + c_n y_n(t)$ is a solution of $L(y) = c_1 f_1(t) + c_2 f_2(t) + \dots + c_n f_n(t)$

In order to apply this, we need the non-homogeneous principle.

Theorem 5 (Non-Homogeneous Principle). $y(t) = y_h(t) + y_p(t)$

What this basically boils down to is making educated guesses in order to identify the form of the particular solution, as well as eventually the particular solution itself. Once the particular and homogeneous solutions are identified, add them to determine the solution. The following table may help identify common formats and solution types.

	$f(t)$	$y_p(t)$
1	k	A_0
2	$P_n(t)$	$A_0(t)$
3	Ce^{kt}	$A_0 e^{kt}$
4	$C\cos(\omega t) + D\sin(\omega t)$	$A_0\cos(\omega t) + B_0\sin(\omega t)$
5	$P_n(t)e^{kt}$	$A_n(t)e^{kt}$
6	$P_n(t)\cos(\omega t) + Q_n(t)\sin(\omega t)$	$A_n(t)\cos(\omega t) + B_n(t)\sin(\omega t)$
7	$Ce^{kt}\cos(\omega t) + De^{kt}\sin(\omega t)$	$A_0 e^{kt}\cos(\omega t) + B_0 e^{kt}\sin(\omega t)$
8	$P_n(t)e^{kt}\cos(\omega t) + Q_n(t)e^{kt}\sin(\omega t)$	$A_n(t)e^{kt}\cos(\omega t) + B_n(t)e^{kt}\sin(\omega t)$

Table 2: Guesses for Particular Solutions

- $P_n(t), Q_n(t), A_n(t), B_n(t) \in \mathbb{P}$
- $A_0, B_0 \in \mathbb{P}_0 \equiv \mathbb{R}$
- $k, \omega, C, D \in \mathbb{R}$
- In [4] and [6] − [8] both terms must be in y_p even if only one term is present in $f(t)$.

If any term or terms of y_p is found in y_h, multiply the term by t or t^2 to eliminate the duplication.

12.6 Variation of Parameters

We've already used variation of parameters to find the solutions of $y' + p(t)y = f(t)$. This same strategy can be applied to second order equations in the form:

$y'' + p(t)y' + q(t)y = f$

To apply this method, follow these steps.

13

14.3.1 Interpreting Non-Real Eigenvalues

$\begin{bmatrix} \vec{\mathbf{x}}_r \\ \vec{\mathbf{x}}_i \end{bmatrix} = e^{\alpha t}\begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix}\begin{bmatrix} \vec{\mathbf{p}} \\ \vec{\mathbf{q}} \end{bmatrix}$

- The first variable defines the expansion.
  - If $\alpha > 0 \to$ Growth without bound.
  - If $\alpha < 0 \to$ Decay to 0.
  - If $\alpha = 0 \to$ Period solutions.
- The second defines rotation.
  - Counterclockwise for $\beta > 0$
  - Clockwise for $\beta < 0$
- The third defines tilt and shape.

14.4 Stability and Linear Classification

A constant solution $\vec{\mathbf{x}} \equiv \vec{\mathbf{c}}$ is called an equilibrium solution. An equilibrium solution in the phase plane is a fixed point.

- If solutions remain close and tend to $\vec{\mathbf{c}}$ as $t \to \infty$ we call this asymptotically stable.
- If solutions are neither attracted nor repelled, we call this neutrally stable.
- If other, it is unstable.

14.5 Parameter Plane

14.6 Possibilities in the Parameter Plane

We have to consider a couple different possibilities.

1. **Real Distinct Eigenvalues** ($\Delta > 0$)

   When $\Delta = (\text{Tr}(A))^2 - 4|A| > 0$ we have real eigenvalues $\lambda_1 \neq \lambda_2$ with corresponding linearly independent eigenvectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ with general solution

   $\vec{\mathbf{x}} = c_1 e^{\lambda_1 t}\vec{\mathbf{v}}_1 + c_2 e^{\lambda_2 t}\vec{\mathbf{v}}_2$

12 Higher Order Linear Differential Equations

$$m\ddot{x} + b\dot{x} + kx = f(t) \tag{23}$$

12.1 Harmonic Oscillators

12.1.1 The Mass-Spring System

Consider an object with mass m on a table that is attached to a spring attached to wall. When the object is moved by an external force, we can model its behavior using Newton's Second Law of Motion: $F = m\ddot{x}$ where F is the sum of the forces acting on the object.

We have three different types of forces:

- **Restoring Force:** The restorative force of a spring is $\propto$ the amount of stretching/compression: $F_{\text{restoring}} = -kx$
- **Damping Force:** We also assume that friction exists, and therefore a damping force $\propto$ the velocity of the object: $F_{\text{damping}} = -b\dot{x}$ Where damping constant $b > 0$ and small for slick surfaces.
- **External Force:** We also allow for an external force to drive the motion: $F_{\text{external}} = f(t)$

Thus we get our equation for a Simple Harmonic Oscillator:

$m\ddot{x} + b\dot{x} + kx = f(t)$

- Constants $m > 0, k > 0, b > 0$
- When $b = 0$, the motion is called undamped. Otherwise it is damped.
- if $f(t) = 0$, the equation is homogeneous and the motion is called unforced, undriven, or free. Otherwise it is forced, or driven.

12.1.2 Solutions

When we say solution, we are referring to a solution that gives us x, in other words, the position of the mass at any given time t as a function of t. Due to the inherent nature of derivatives, this may or may not have undetermined constants (often denoted as $[c_1, c_2, \dots, c_n]$) as will be set by initial values given (similar to first order differential equations).

Later we will determine how to solve these equations fully, however a quick answer can be found by applying the following formulas. After learning

1. Find two linearly independent solutions of the second order differential equation $y'' + p(t)y' + q(t)y = f(t)$ this having the general solution $y_h(t) = c_1 y_1(t) + c_2 y_2(t)$
2. To find the particular solution, take $y_h(t) = c_1 y_1(t) + c_2 y_2(t)$ and swap constants to get $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$ where v_1 and v_2 are unknown functions.
3. We find v_1 and v_2 by substituting our new equation into our first. Differentiating by the product rule we get $y_p'(t) = v_1 y_1' + v_2 y_2' + v_1' y_1 + v_2' y_2$
4. Before we calculate y_p'' we choose an auxiliary condition, that v_1 and v_2 satisfy $v_1' y_1 + v_2' y_2 = 0$ where we get $y_p' = v_1 y_1' + y_2' v_2$
5. Differentiating again we get $y_p''(t) = v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2'$
6. We wish to get $L(y) = y'' + py' + qy = f$ Substituting for what we have solved for gives $v1y_1' + v_2' y_2' = 0$
7. We now have two equations for our two unknowns. $\begin{cases} y_1 v_1' + y_2 v_2' = 0 \\ y_1' v_1' + y_2' v_2' = f \end{cases}$
8. Solve the system of equations and insert.

Another method is to use Cramer's Rule (**??**) where

$v_1' = \dfrac{\begin{bmatrix} 0 & y_2 \\ f & y_2' \end{bmatrix}}{\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}}$ and $v_2' = \dfrac{\begin{bmatrix} y_1 & 0 \\ y_1' & f \end{bmatrix}}{\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}}$

The denominator in this case is the Wronskian. It will not be zero because both y_1 and y_2 are linearly independent. Integrate these to find v_1 and v_2.

13 Linear Transformations

Vectors that aren't rotated by linear transformations, but are only scaled or flipped are called eigenvectors.

Theorem 6 (Eigenvalues and Eigenvectors). *Let $T : \mathbb{V} \to \mathbb{V}$ be a linear transformation. A scalar λ is an eigenvalue of T is there is a nonzero vector $\vec{\mathbf{v}} \in \mathbb{V}$ such that $T(\vec{\mathbf{v}}) = \lambda\vec{\mathbf{v}}$.*

Such a nonzero vector $\vec{\mathbf{v}}$ is called an eigenvector of T corresponding to λ.

If the linear transformation T is regenerated by an $n \times n$ matrix A where $\mathbb{V} = \mathbb{R}^n$ and $T(\vec{\mathbf{v}}) = A\vec{\mathbf{v}}$, then A and $\vec{\mathbf{v}}$ are characterized by the equation $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$.

The signs of the eigenvalues direct the trajectory behavior in the phase 14 portrait.

We can label the eigendirections fast or slow based on the magnitude of the eigenvalues. Whichever it is, the trajectories are parallel to fast and perpendicular to slow.

Three possibilities

- Attracting Node ($\lambda_1 < \lambda_2 < 0$)
- Repelling Node ($0 < \lambda_1 < \lambda_2$)
- Saddle Point ($\lambda_1 < 0 < \lambda_2$)

2. **Complex Conjugate Eigenvalues** ($\Delta < 0$)

   When $\Delta = (\text{Tr}(A))^2 - 4|A| < 0$ we get non-real eigenvalues.

   $\lambda_{1,2} = \alpha \pm \beta i$

   where $\alpha = \frac{\text{Tr}(A)}{2}$ and $\beta = \sqrt{-\Delta}$. α and β are real. The real solutions are given by:

   $\begin{cases} \vec{\mathbf{x}}_r = e^{\alpha t}(\cos(\beta t)\vec{\mathbf{p}} - \sin(\beta t)\vec{\mathbf{q}}) \\ \vec{\mathbf{x}}_i = e^{\alpha t}(\sin(\beta t)\vec{\mathbf{p}} + \cos(\beta t)\vec{\mathbf{q}}) \end{cases}$

   For complex eigenvalues stability behavior depends on the sign of α.

   - Attracting Spiral ($\alpha < 0$)
   - Repelling Spiral ($\alpha > 0$)
   - Center ($\alpha = 0$)

3. **Borderline Case: Zero Eigenvalues** ($|A| = 0$) If one eigenvalue is zero we get a row of non-isolated fixed points in the eigendirection associated with the eigenvalues, and the phase plane trajectories are all straight lines in direction of other eigenvector.

   If two eigenvalues are zero, there is only one eigenvector, along which we have a row of non-isolated fixed points. Trajectories from any other point in the phase plane must be parallel to the one eigenvector in the direction specified by the system.

4. **Borderline Case: Real Repeated Eigenvalues** ($\Delta = 0$)

   In this situation we have two cases to contend with.

   (a) *Degenerate Node:* If λ has one linearly independent eigenvector we call it degenerate. The sign of λ gives its stability.

the methods given ahead, be sure to come back and determine how these solutions were determined.

Given Equation: $m\ddot{x} + kx = 0$

$x(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

This gives us one form of the solution, however we can also find an alternate form:

$x(t) = A\cos(\omega_0 t - \delta)$

Where

- Amplitude A and phase angle δ (radians) are arbitrary constants determined by initial conditions.
- The motion has circular frequency $\omega_0 = \sqrt{\frac{k}{m}}$ (radians) per second, and a natural frequency $f_0 = \frac{\omega_0}{2\pi}$
- The period T (seconds) is $2\pi\sqrt{\frac{m}{k}}$
- The above solution is a horizontal shift of $A\cos(\omega_0 t)$ with phase shift $\frac{\delta}{\omega_0}$

To convert between the two forms, apply the following formulas.

$\begin{cases} A = \sqrt{c_1^2 + c_2^2} \\ \tan\delta = \frac{c_2}{c_1} \end{cases} \begin{cases} c_1 = A\cos\delta \\ c_2 = A\sin\delta \end{cases}$

To solve the Mass-Spring System with both damping and forcing as given by the following equation:

$m\ddot{x} + b\dot{x} + kx = F_0\cos(\omega_f t)$

we can apply the following formula. (Note, some concepts are explained later in the text, refer back if needed)

1. $x_h(t)$ has three possible solutions. See (**??**).
2. $x_p(t)$ can be assumed as $A\cos(\omega_f t) + B\sin(\omega_f t)$ See (**??**).
3. $\omega_0 = \sqrt{\frac{k}{m}}$
4. $A = \frac{m(\omega_0^2 - \omega_f^2)F_0}{m^2(\omega_0^2 - \omega_f^2)^2 + (b\omega_f)^2}$
5. $B = \frac{b\omega_f F_0}{m^2(\omega_0^2 - \omega_f^2)^2 + (b\omega_f)^2}$

As you can see, this is a pain. Values A and B in particular are tedious to calculate. Despite this, as you'll see later, these methods can be easier than solving by hand.

11 To compute these eigenvalues and eigenvectors, follow the following steps[3].

1. Write the characteristic equation $|A - \lambda I| = 0$
2. Solve the characteristic equation for the eigenvalues.
3. For each eigenvalue, find the eigenvector by solving $(A - \lambda_i I)\,\vec{\mathbf{v}}_i = 0$

As you'd imagine, once the size of a matrix becomes larger than 2 or 3, these steps are tedious and long. Computers to the rescue!

13.1 Special Cases

Some special cases to watch out for:

- **Triangular Matrices:** The eigenvalues of a triangular matrix (upper *or lower*) appear on the main diagonal.
- **2×2 Matrices:** The eigenvalues can be determined with $\lambda^2 - (Tr^4(\mathbf{A}))\lambda + |\mathbf{A}| = 0$
- **3×3 Matrices:** Similarly: $\lambda^3 - \lambda^2\text{Tr}(\mathbf{A}) - \lambda\frac{1}{2}\left(\text{Tr}(\mathbf{A}^2) - \text{Tr}^2(\mathbf{A})\right) - \det(\mathbf{A}) = 0$

13.2 Eigenspaces

The set of all eigenvectors belonging to an eigenvalues λ together with the zero vector form a subspace of $\mathbb{R}^n$ called the eigenspace.

Theorem 7 (Eigenspaces). *For each eigenvalue λ of a linear transformation $T : \mathbb{V} \to \mathbb{V}$, the eigenspace $\mathbb{E}_\lambda = \{\vec{\mathbf{V}} \in \mathbb{V} \,|\, T(\vec{\mathbf{v}}) = \lambda\vec{\mathbf{v}}\}$ is a subspace of $\mathbb{V}$.*

Theorem 8 (Distinct Eigenvalue). *Let $\mathbf{A}$ be an $n \times n$ matrix. If $\lambda_1, \lambda_2, \dots, \lambda_p$ are distinct eigenvalues with corresponding eigenvectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$, then $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n\}$ is a set of linearly independent vectors. In other words, if each eigenvalue has one associated eigenvector, than that set of eigenvectors is linearly independent.*

[3]Note, the same exact steps are followed even if we have λ to be in terms of i. The only exception is that we are no longer in any $\mathbb{R}^n$ space, and therefore there will be no *real* eigenspace (See (**??**))

[4]Where $Tr(\mathbf{A})$ is the Trace of a matrix, i.e. the sum of the main diagonal.

   (b) *Star Node:* If λ has two linearly independent eigenvectors we call it an attracting or repelling star node. The sign of λ gives its stability.

   In both cases, the sign of λ gives its stability.

   - If $\lambda > 0$, trajectories go to infinity, parallel to $\vec{\mathbf{v}}$.
   - If $\lambda < 0$, trajectories approach the origin parallel to $\vec{\mathbf{v}}$.
   - If $\lambda = 0$, there exists a line of fixed points at the eigenvector.

15 Non-Linear Systems

15.1 Properties of Phase Plane Trajectories in Non-Linear 2×2 Systems

1. When uniqueness holds, phase plane trajectories cannot cross.
2. When the given functions f and g are continuous, trajectories are continuous and smooth.

15.2 Equilibria

Phase Portraits can have more than one, or none at all. To find a system's equilibria, solve x' and y' simultaneously.

15.3 Nullclines

Nullclines in this case are the same as before.

15.4 Limit Cycle

A limit cycle is a closed curve (representing a periodic solution) to which other solutions tend by winding around more and more closely from either inside or outside.

16 Linearization

Theorem 9 (Jacobian). *For a given system of equations:*

$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$

12.1.3 Phase Planes

For any autonomous second order differential equation

$\ddot{x} = F(x, \dot{x})$

the phase plane is the two dimensional graph with x and $\dot{x}$ axes (which are the position and velocity respectively)[2]. This phase plane has a vector field with direction given by

$\begin{cases} H \to \frac{dx}{dt} = \dot{x} \\ V \to \frac{d\dot{x}}{dt} = \ddot{x} \end{cases}$

Trajectories can be formed by parametrically combining the vectors into a path. A graph showing these trajectories is called a phase portrait.

The differential equation is also equivalent to the system of equations:

$\begin{cases} \dot{x} = y \\ \dot{y} = \ddot{x} = f(t) - \frac{k}{m}x - \frac{b}{m}y \end{cases}$

The biggest advantage with phase portraits is that is allows the user to solve the differential equation graphically, and not numerically. This can be much easier if done correctly.

12.2 Properties and Theorems

For the linear homogeneous, second-order differential equation

$y'' + p(t)y' + q(t)y = 0$

with p and q being continuous functions of t, there exists a two-dimensional vector space of solutions.

Rewriting the above equation gives us

$y''(t) \equiv f(t, y, y') = -p(t)y' - q(t)y = 0$

which gives us the existence and uniqueness theorem for the second order equation.

Theorem 2 (Existence and Uniqueness). *Let $p(t)$ and $q(t)$ be continuous on a, b containing t_0. For any A and B in $\mathbb{R}$, there exists a unique solution $y(t)$ defined on (a, b) to the IVP $y'' + p(t)y' + q(t)y = 0, y(t_0) = A, y'(t_0) = B$*

A basis exists for the general second order equation.

Theorem 3 (Solution Space). *The solution space S for a second order homogeneous differential equation has a Dimension of 2.*

For any linear second order homogeneous differential equation on (a, b),

$y'' + p(t)y' + q(t)y = 0$

[2]This concept of a phase plane is identical to the one introduced in (**??**) with the exception of $\dot{x}$ replacing y.

13.3 Properties of Eigenvalues

Let $\mathbf{A}$ be an $n \times n$ matrix.

- λ is an eigenvalue of A if and only if $|\mathbf{A} - \lambda\mathbf{I}| = 0$
- λ is an eigenvalue of A if and only if $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$ has a non-trivial solution.
- $\mathbf{A}$ has a zero eigenvalue if and only if $|\mathbf{A}| = 0$
- $\mathbf{A}$ and $\mathbf{A}^T$ have the same characteristic polynomials and eigenvalues.

13.4 The Mind-Blowing Part

Remember Characteristic Roots (**??**)? Well, they are *identical* to eigenvalues as is evidenced below.

Given the linear second order differential equation:

$y'' - y' - 2y = 0$

we know that it has a characteristic equation of

$r^2 - r - 2 = (r - 2)(r + 1) = 0$

with roots of

$[r_1, r_2] \begin{cases} 2 \\ -1 \end{cases}$

which creates the general solution of

$y = c_1 e^{2t} + c_2 e^{-t}$

In Section **??** we saw that we can write a second order differential equation as a system of equations:

$\begin{cases} \dot{x} = y' \\ \dot{y} = 2y + y' \end{cases}$

which has the matrix form $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$:

$\vec{\mathbf{x}} = \begin{bmatrix} y \\ y' \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$

The characteristic equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$ for this matrix $\mathbf{A}$ is $\lambda^2 - \lambda - 2 = 0$ which has the same eigenvalues as our original equation has characteristic roots.

13.4.1 Properties of Linear Homogeneous Differential Equations with Distinct Eigenvalues

For the differential equation $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ with distinct eigenvalues, the following properties apply.

Type	Eigenvalues	Linearized System 15		Nonlinear System	
		Geometry	Stability	Geometry	Stability
Real Distinct Roots	$\lambda_1 < \lambda_2 < 0$	Attracting Node	Asymptotically Stable	Attracting Node	Asymptotically Stable
	$0 < \lambda_2 < \lambda_1$	Repelling Node	Unstable	Repelling Node	Unstable
	$\lambda_1 < 0 < \lambda_2$	Saddle	Unstable	Saddle	Unstable
Real Repeated Roots	$\lambda_1 = \lambda_2 < 0$	Attracting Star of Degenerate Node	Asymptotically Stable	Attracting Node or Spiral	Asymptotically Stable
	$\lambda_1 = \lambda_2 > 0$	Repelling Star or Degenerate Node	Unstable	Repelling Node or Spiral	Unstable
Complex Conjugate Roots	$\alpha > 0$	Repelling Spiral	Unstable	Repelling Spiral	Unstable
	$\alpha < 0$	Attracting Spiral	Asymptotically Stable	Attracting Spiral	Asymptotically Stable
	$\alpha = 0$	Center	Stable	Center or Spiral	Uncertain

Table 3: Table of Behavior Based on the System's Jacobian Matrix Eigenvalues

where f and g are twice differentiable, the linearized system at an equilibrium point (x_e, y_e) translated by $u = x - x_e$ and $v = y - y_e$ is

$$\begin{bmatrix} u \\ v \end{bmatrix}' = J(x_e, y_e) \text{ where } J(x_e, y_e) = \begin{bmatrix} f_x(x_e, y_e) & f_y(x_e, y_e) \\ g_x(x_e, y_e) & g_y(x_e, y_e) \end{bmatrix} \tag{26}$$

for which p and q are continuous on (a, b), any two linearly independent solutions $\{y_1, y_2\}$ form a basis of the solutions space S, and every solution y on (a, b) can be written as

$y(t) = c_1 y_1(t) + c_2 y_2(t) \to (c_1, c_2) \in \mathbb{R}$

To generalize we can apply the same principle to nth order differential equations.

Theorem 4 (Existence and Uniqueness for nth Order Differential Equations). *Let $p_1(t), p_2(t), \dots, p_n(t)$ be continuous functions on (a, b) containing t_0. For any initial values $A_0, A_1, \dots, A_{n-1} \in \mathbb{R}$, there exists a unique solution $y(t)$ to the IVP $y''(tp_1(t)y^{n-1}(t)) + p_1(t)y^{n-1}(t) + p_2(t)y^{n-2}(t) + \dots + p_n(t)y(t) = 0$*

$$y(t_0) = A_0, y'(t_0) = A_1, \dots, y^{n-1}(t_0) = A_{n-1}$$

For nth order differential equations, our solution space theorem (**??**) applies, just replace the term "2" and "second" with "n" and "nth".

12.3 Roots

If given a second order equation in the form $a\ddot{y} + b\dot{y} + cy = 0$, we can use our previous definition of a first order differential equation to find an easier method of solving. At its core, this method consists of converting our given second order differential equation and converting it into a quadratic equation, using which we can solve for the homogeneous solution.

$$a\ddot{y} + b\dot{y} + cy = 0 \Leftrightarrow ar^2 + br + c = 0 \tag{24}$$

The resulting equation is called the characteristic equation. Solutions to this equation are called characteristic roots. Due to the nature of quadratic equations, there are three different possibilities for the solution:

- Two distinct real roots or zeros
- One real root (a double root)
- Two imaginary roots

These are summarized as follows.

These methods allow us to generalize for higher order differential equations and find solutions that would be otherwise impossible.

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- The domain of the linear transformation is a vector space of vector functions.
- The solution set is also a vector space of vector functions.
- The eigenspace for each eigenvalue is a one dimensional line in the direction of a vector in $\mathbf{R}^n$.

14 Linear Systems of Differential Equations

To define the linear first order differential equations system:

An n-dimensional first order differential equations system on an open interval I is one that can be written as a matrix vector equation.

$$\vec{\mathbf{x}}'(t) = A(t)\vec{\mathbf{x}}(t) + \vec{\mathbf{f}}(t) \tag{25}$$

- $A(t)$ is an $n \times n$ matrix of continuous functions on I.
- $\vec{\mathbf{f}}(t)$ is an $n \times 1$ vector of continuous functions on I.
- $\vec{\mathbf{x}}(t)$ is an $n \times 1$ solution vector.
- If $\vec{\mathbf{f}}(t) \equiv 0$, the system is homogeneous.

14.1 Graphical Methods

We use the phase plane from before to accurately represent these systems.

14.1.1 Nullclines

The v nullcline is the set of all points with vertical slope which occur on the curve obtained by solving $x' = f(x, y) = 0$ The h nullcline is the same except with horizontal slope and is found with $y' = f(x, y) = 0$ At the intersection we get a fixed equilibrium point.

14.1.2 Eigenvalues

Eigenvalues play a large role in phase planes as well. For an autonomous and homogeneous system of differential linear system of equations:

- Trajectories are toward or away based on the sign of the eigenvalue.
- Along each eigenvector is the separatrix that seperates different curves.