Fourier Series Notes

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1 Review

1.1 Series

$$\sum_{n=0}^{\infty} r^n = 0 \qquad 0 < r < 1$$

1.2 Ordinary Differential Equations

Theorem 1 (Picard's Theorem)**.** *If an nth order ODE has n initial conditions, then the solutions exists in a neighborhood of initial conditions. The solution is unique when*

$$\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$$

TL;DR An nth order equation needs n initial conditions.

TODO: Flesh out ODE review

1.3 Calculus 3

TODO: Flesh out calc 3 review

2 Fourier Series

Theorem 2 (Fourier Series Expansion on $-\pi \leq x \leq \pi$)**.** *If we change basis of functions we can write them as a Fourier Series with the following equation on the interval* $[-\pi, \pi]$*.*

$$a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \tag{1}$$

These coefficients are as follows for $[-\pi, \pi]$*.*

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

This function is defined everywhere and is 2π periodic.

Theorem 3 (Fourier Series Expansion on $-L \leq x \leq L$)**.** *Same equation, but different coefficients.*

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$
$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{nx\pi}{L}\right)\, dx$$
$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{nx\pi}{L}\right)\, dx$$

2.1 Piecewise Continuity

A function is **Piecewise Smooth** on some interval if it can be broken up into pieces such that in each piece the function is continuous and its derivative is also continuous. The entire function does not need to be continuous, but can only have a finite number of jump discontinuities[1].

2.2 Convergence for Fourier Series

Theorem 4 (Fourier's Theorem)**.** *If $f(x)$ is piecewise smooth on the interval $-L \leq x \leq L$, then the Fourier series of $f(x)$ converges*

1. *to the periodic extension of $f(x)$, where the periodic extension is continuous;*
2. *to the average of the two limits, usually*

$$\frac{1}{2}[f(x_+) + f(x_-)]$$

where the periodic extension has a jump discontinuity.

[1] Jump discontinuities are when the left and right limits both exist and are unequal.

2.3 Cosine and Sine Series

2.3.1 Sine Series

An odd function is defined as $f(-x) = f(x)^2$. We can calculate the Fourier Series of odd functions.

$$
\begin{aligned}
a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)\, dx = 0 \\
a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{nx\pi}{L}\right)\, dx = 0 \\
b_n &= \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{nx\pi}{L}\right)\, dx
\end{aligned}
$$

However this doesn't happen often, and instead we usually just find the odd extension of a function.

$$
\begin{aligned}
f(x) &\sim \sum_{n=1}^{\infty} \left[B_n \sin\left(\frac{nx\pi}{L}\right) \right] \\
B_n &= \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{nx\pi}{L}\right)\, dx
\end{aligned}
$$

This is the Fourier sine series on the interval $0 \leq x \leq L$.

2.3.2 Cosine Series

We can apply similar methods to even functions, where $f(-x) = f(x)$.

$$
\begin{aligned}
f(x) &\sim \sum_{n=0}^{\infty} \left[a_n \cos\left(\frac{nx\pi}{L}\right) \right] \\
a_0 &= \frac{1}{L} \int_{0}^{L} f(x)\, dx \\
a_n &= \frac{2}{L} \int_{0}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \\
b_n &= 0
\end{aligned}
$$

Similarly, we can find the even extension of a given function.

[2] In other words, reflected about the y axis

$$f(x) \sim \sum_{n=0}^{\infty} \left[A_n \cos\left(\frac{nx\pi}{L}\right) \right]$$
$$A_0 = \frac{1}{L} \int_0^L f(x)\, dx$$
$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{nx\pi}{L}\right)\, dx$$

2.3.3 Even and Odd Parts

Theorem 5. *The Fourier series of $f(x)$ equals the Fourier sine series of $f_o(x)$ plus the Fourier cosine series of $f_e(x)$, where $f_e(x) = \frac{1}{2}[f(x) + f(-x)]$, and $f_o(x) = \frac{1}{2}[f(x) - f(-x)]$.*