Chaos and Collatz - A Simple Map

APPM 3010, University of Colorado, Boulder

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Zoë Farmer

CU Boulder Applied Math

A Brief Introduction to the Collatz Problem

Let x be an arbitrary positive number, i.e. $x \in \mathbb{Z}$, $x > 0$. Define $f : \mathbb{Z} \to \mathbb{Z}$ as the following.

$$f(x) = \begin{cases} x/2 & x \equiv 0 \mod 2 \\ 3x + 1 & x \equiv 1 \mod 2 \end{cases}$$

Now define the sequence C_x as the iteration of this function,

$$C_{x,n+1} = f(C_{x,n})$$

The Collatz Conjecture states that for any input number x, C_x will go to one as n goes to infinity. In limit form,

$$\lim_{n \to \infty} C_{x,n} = 1$$

Professional Opinions

Paul Erdös once said, “Mathematics is not yet ready for such problems.”

Visualizing the Problem

- Collatz was famous for visualizing this as a directed graph
- Two different ways
- Top Down - Pick a number and iterate
- Bottom up - Make branches

Top Down Algorithm

1	Initialize an array of edges called E.
2	Let $i = 2$.
3	Find the corresponding Collatz Sequence for i, C_i.
4	Add edges to E of the form $(C_{i,n}, C_{i,n+1})$.
5	Let $i = i + 1$ and go to 3.

Table: Top Down Collatz Graph Algorithm

Top Down Graph

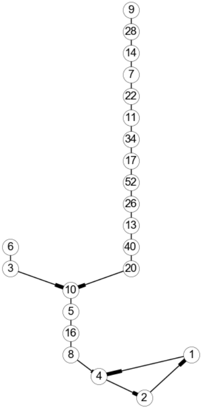


Figure: Directed Graph of the Collatz Sequence from 1 to 20

Bottom Up Algorithm

1	Initialize an array of edges called E.
2	Initialize an array of "start nodes", called S.
3	Set the depth D.
4	Initialize a queue, Q, filled with (S_i, D).
5	Get (s, d) from the queue. If $\|Q\| = 0$, break.
6	Set $x_1 = 2s$, and $x_2 = (2x - 1)/3$.
7	Add (x_1, s) to E.
8	If $s \equiv 4 \mod 6$, add (x_2, s) to E, else do nothing.
9	If $d = 0$, go to 12, else continue.
10	Add $(x_1, d - 1)$ to Q.
11	If $s \equiv 4 \mod 6$, add $(x_2, d - 1)$ to Q, else do nothing.
12	Go to 5

Table: Bottom Up Collatz Graph Algorithm

Bottom Up Graph

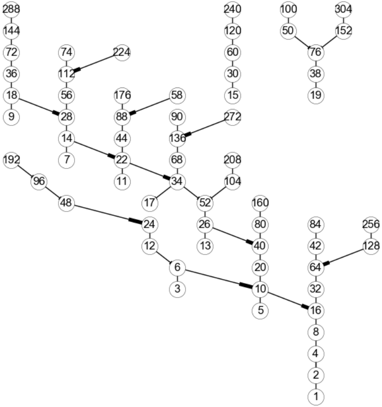


Figure: Directed Graph of the Collatz Sequence from 1 to 10

Let's scale things up a bit

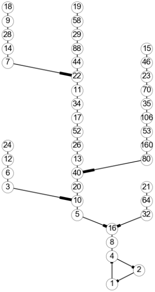


Figure: Directed Graph of the Collatz Sequence from 1 to 25

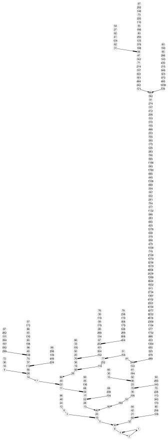


Figure: Directed Graph of the Collatz Sequence from 1 to 100

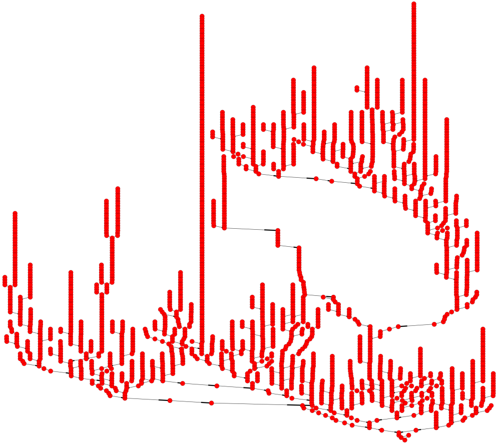


Figure: Directed Graph of the Collatz Sequence from 1 to 1000

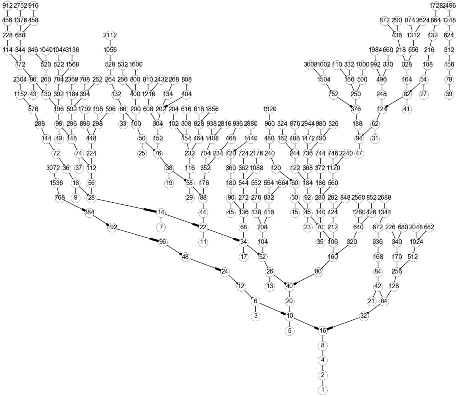
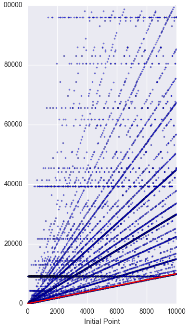
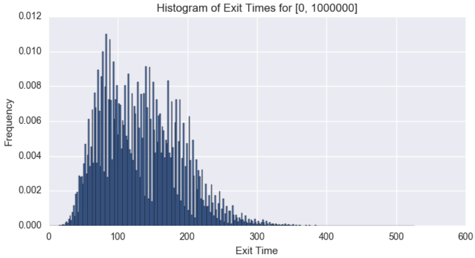


Figure: Directed Graph of the Collatz Sequence from 1 to 50

What else can we look for?

- Maximal point achieved during iteration
- Histogram of exit times for the first million numbers





Longest Chain

Longest Collatz sequence

Problem 14

The following iterative sequence is defined for the set of positive integers:

$n \to n/2$ (n is even)
$n \to 3n + 1$ (n is odd)

Using the rule above and starting with 13, we generate the following sequence:

$$13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$$

It can be seen that this sequence (starting at 13 and finishing at 1) contains 10 terms. Although it has not been proved yet (Collatz Problem), it is thought that all starting numbers finish at 1.

Which starting number, under one million, produces the longest chain?

NOTE: Once the chain starts the terms are allowed to go above one million.

Figure: Problem 14 on ProjectEuler

This results in the number 837799 which has a chain 526 entries long.

Through a Chaos Theory Lens

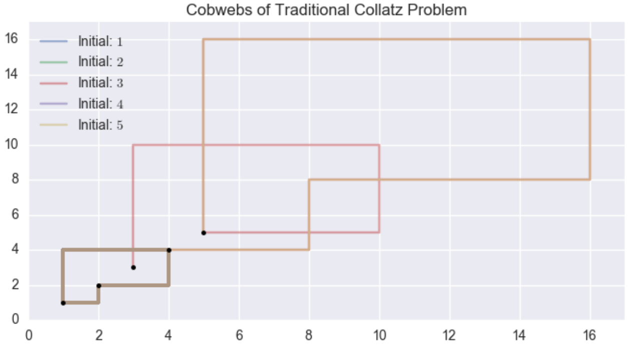


Figure: Cobweb of Several Collatz Sequences

Stable Orbit?

We seem to have a super-stable attractor at 1, which matches our Collatz Conjecture.

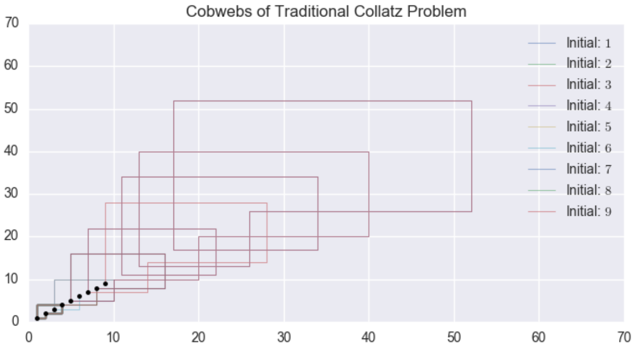


Figure: Cobweb of Several More Collatz Sequences

But is it Chaotic?

Although no universally accepted mathematical definition of chaos exists, a commonly used definition originally formulated by Robert L. Devaney says that, to classify a dynamical system as chaotic, it must have these properties:

1. it must be sensitive to initial conditions
2. it must be topologically mixing
3. it must have dense periodic orbits

Sensitive Dependence?

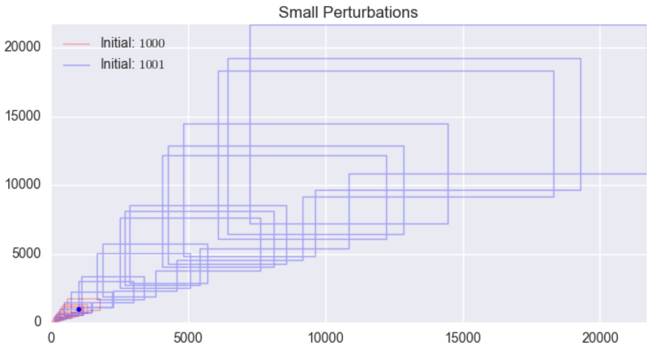


Figure: Initial Sensitivity

Dense Orbits?

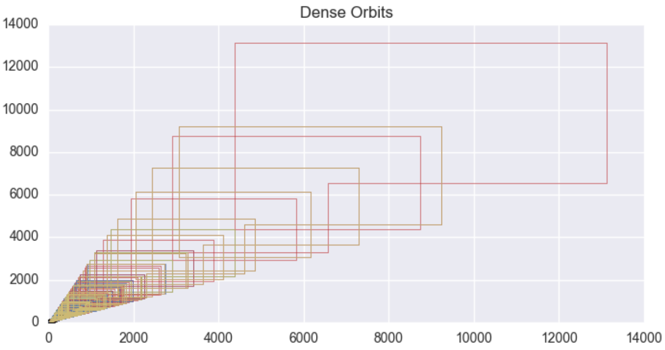


Figure: Dense Orbits from 1 to 300

We can try to find these, but we immediately run into a problem.

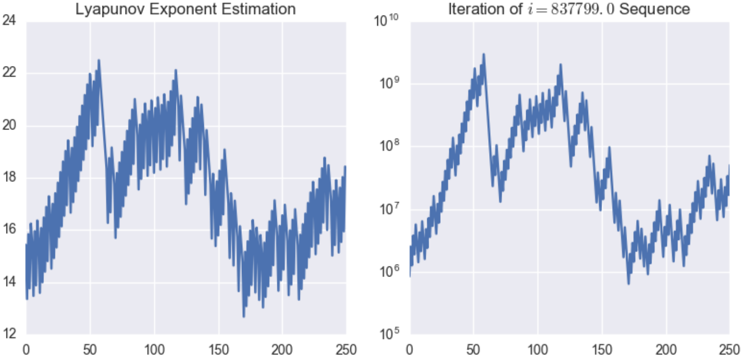
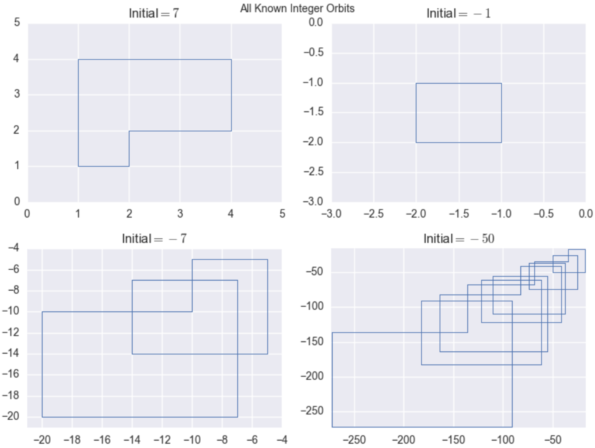


Figure: Lyapunov Exponent Estimation

Generalization - To All Integers

The same rules apply.



Where do all Integers end up??

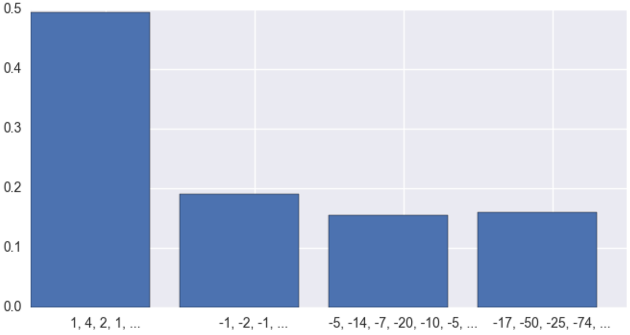
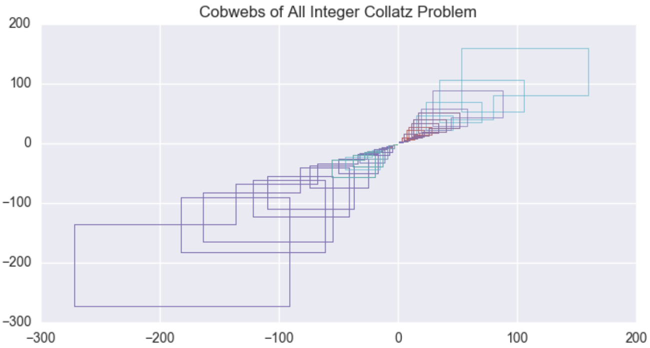


Figure: Histogram of Orbits for all Points from $-10,000$ to $10,000$

Large Cobweb



Generalization - To all Real Numbers

By noting that the two operations alternate, we can define a function accordingly.

$$f(x) = \frac{1}{2}x\cos^2\left(\frac{\pi}{2}x\right) + (3x+1)\sin^2\left(\frac{\pi}{2}x\right)$$

A More Accurate Cobweb

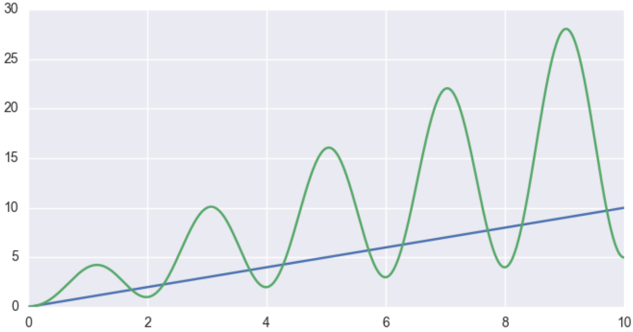


Figure: Collatz Conjecture Real Extension

Dense Sampling from 1 to 5

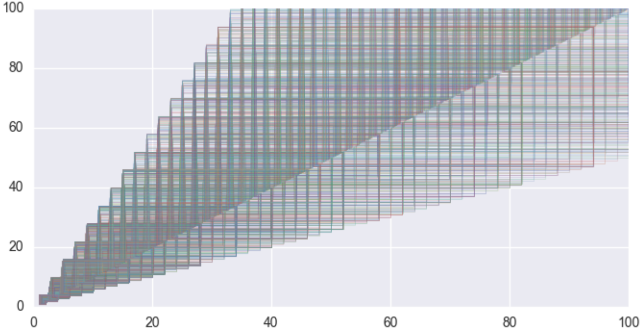


Figure: Orbits on the Real Collatz Map

Example Orbits

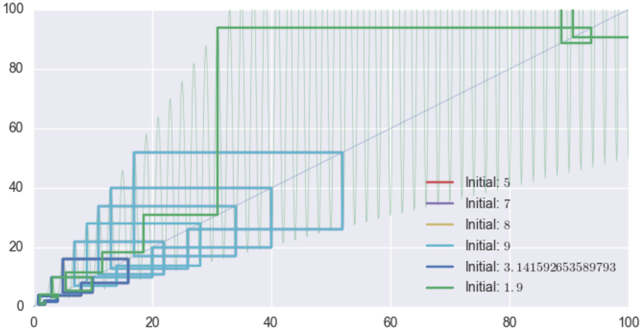


Figure: Orbits on the Real Collatz Map

Escaping?

Let's find orbits that don't escape.

1	Initialize a space of possible orbits called $\mathcal{O}$.
2	Set $i = 1$.
3	Find the Collatz Sequence for $\mathcal{O}_i$.
4	If this sequence converges to an orbit, $\mathcal{O}_i$ is stable.
5	Set $i = i + 1$ and go to 3.

Table: Finding Stable Orbits

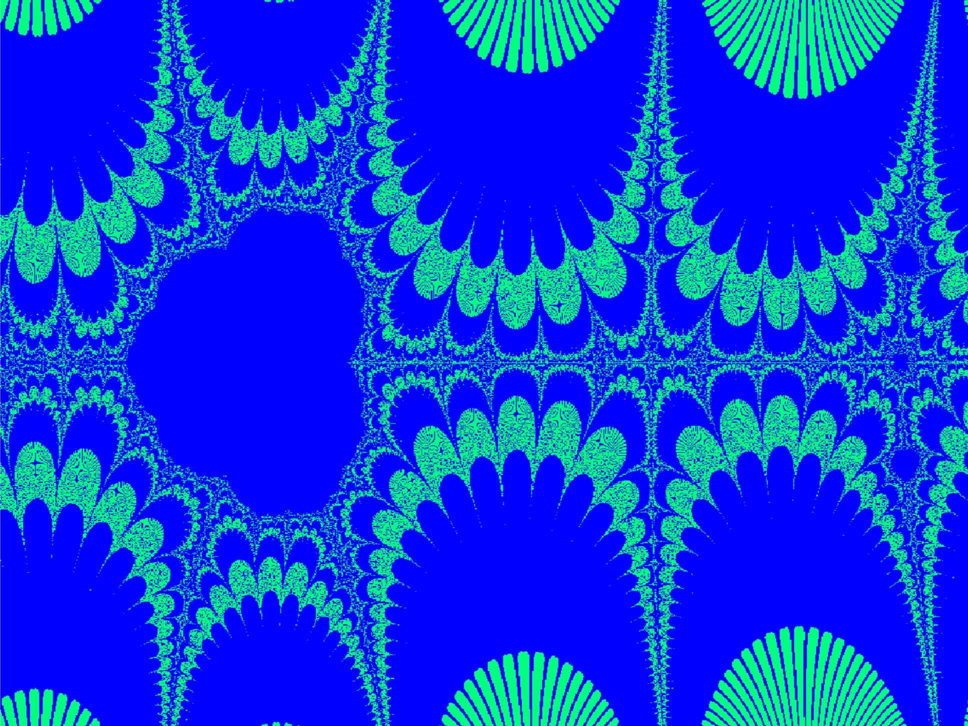
Only 10% of the densely sampled orbits shown previously converge!

Fractals

This is well-behaved for complex numbers as well.

$$f(z) = \frac{1}{2} z \cos^2\left(\frac{\pi}{2} z\right) + (3z+1)\sin^2\left(\frac{\pi}{2} z\right)$$

And since it shows this escaping behavior we can create a fractal in the same way that Mandelbrot, etc. are created.
The next slide shows a portion from $-1 - i$ to $2.5 + i$.



Final Thoughts

- It's a fascinating one-dimensional map
- I would hesitate to call it "chaotic" in the mathematical sense.
- Is the conjecture true? Maybe... As far as I can tell yes.